January 28, 2009 WOCOMAL Varsity Meet

Round 1: Similarity and Pythagorean Theorem

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the second diagonal of a rhombus of side length 18 cm when one diagonal is $15\sqrt{2}\ cm$. Write answer in simple radical form.

2.

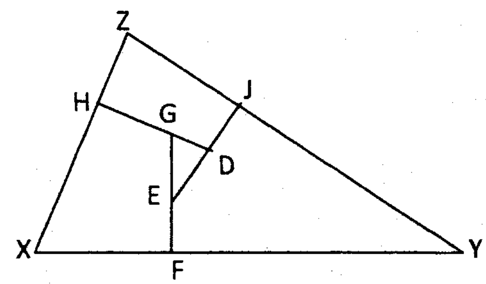
$\overline{GF} \perp \overline{XY}$,

$\overline{EJ} \perp \overline{YZ}$,

$\overline{DH} \perp \overline{XZ}$,

$XZ = 10$, XY=12, YZ=15.

Find $\frac{DE}{DG}$.



3. The medians of a right triangle which are drawn from the vertices of the acute angles are 5 and $2\sqrt{10}$. Find the length of the hypothenuse. Write answer in simple radical form.

ANSWERS

(1 pt.) 1. ______________________ cm

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Doherty, Holy Name, St. John's

January 28, 2009 WOCOMAL Varsity Meet

Round 2: Algebra I

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. In addition to spanning a river which is 50 m wide, a straight bridge has considerable overlap of both banks of the river. One third of the bridge overlaps one bank and one half of the bridge overlaps the other bank. Find the total length of the bridge

2. If the sum of two numbers is twenty and their product is 30, compute the sum of their reciprocals.

3. Find the sum of two positive integers whose product is 24,999,999 and whose positive difference is as small as possible.

ANSWERS

(1 pt.) 1. ______________________m

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Hudson, Mass Academy

January 28, 2009 WOCOMAL Varsity Meet

Round 3: Functions

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. $f(g(x)) = x^3 - 8$. If $f(x) = x^3$, find $g(x)$.

2. If $F(x) = \dfrac{\sqrt{3x-2}}{4}$, where $x \geq \dfrac{2}{3}$. Evaluate $F^{-1}(5)$, if it exists.

3. If $f(x) = \dfrac{\left(3 - \dfrac{4}{x}\right)^2}{9 - \dfrac{16}{x^2}}$ then $g(y) = \dfrac{f(\frac{5}{3} + y) - f\left(\frac{5}{3}\right)}{y}$ can be written in reduced form as $\dfrac{a}{by + c}$.

Find $a + b + c$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Assabet Valley, Bancroft, Holy Name

Round 4: Combinatorics

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. How many ways are there of scoring exactly 70 % on a 10-question true-false test?

2. A basketball coach has 9 first string players, all of whom can play any of these positions (center, forward, and guard) on the five- man team. How many different teams of 1 center 2 forwards, and 2 guards can be formed using any of the 9 players? Teams are considered different if players play different positions even if the same 5 players are involved. Assume no difference between left guard and right guard and no difference between left forward and right forward.

3. A cube of edge 3 has each of its faces divided into 9 squares of side 1 each. Compute how many paths of length 9 there are from one vertex of the cube to the opposite vertex of the cube that are contained exclusively on the edges of the small squares drawn on the surface of the cube.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Quaboag, St. John's, Shrewsbury

January 28, 2009 WOCOMAL Varsity Meet

Round 5: Analytic Geometry

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the value of k so that the equation $x^2+y^2-10x+4y+k=0$ is the equation of a circle of radius $2\sqrt{3}$.

2. If $\frac{x^2}{25}+\frac{y^2}{c^2-9}=1$, under what condition(s) on c will the graph of this equation be a hyperbola?

3. When the equation for the set of all centers of circles which pass through the points $(3,3)$ and $(-6,-2)$ is written in the form $Ax+By+C=0$ where A, B, and C are relatively prime, what is the absolute value of the product ABC?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Bromfield, Nashoba, Westborough

January 28, 2009 TEAM ROUND ANSWER SHEET WOCOMAL Varsity Meet

(2 points each)

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ________________________ minutes

8. $_____________________________

9. ______________________________

School ______________________________

Team # ______________________________

Student Names:

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ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM AND ON THE SEPARATE TEAM ANSWER SHEET (2 points each)

1. If $f(x) = 3x^2 + 2$, $g(x) = 2x + 4$, and $h(x) = \frac{1}{2}x - 3$, *find* $h(f(g(x)))$ in simplified form.

2. How many ordered pairs (x, y) satisfy $(|x|, |y|) = (6 - |x|, 2)$.

3. In a right triangle the hypothenuse is 13 and the altitude to the hypotenuse is 6. Find the sum of the lengths of the legs of the triangle. Write answer in simple radical form.

4. Find the unit digit of the sum: $1! + 2! + 3! + \cdots + 14! + 15!$

5. Find the greatest integer value for $\frac{CD}{E}$, if $\frac{x^2}{9-C} + \frac{y^2}{5-C} = 1$ must be a hyperbola, $\frac{x^2}{8-D} + \frac{y^2}{12-D} = 1$ must be an ellipse, and $\frac{x^2}{2(E+4)} + \frac{y^2}{4E} = 1$ must be a circle.

6. If $f(x+1) = \frac{f(x) - f(x-1) + 10}{f(x-2)}$, *and* $f(1) = 2f(2) = 4f(3) = 8$, *find* $f(6)$.

7. A digital clock displays three digits between 2:00 and 4:00. For how many minutes during this period do two of the digits displayed sum to the third digit?

8. Find the total value of all combinations of 3 coins you can make using only pennies, nickels, dimes, and quarters.

9. All the words in the snowman language consists of exactly seven letters formed from the "word units" {s, no, wm, an} arranged in some order, where the paired letters cannot be separated. For example, "snonono" and "sssanwm" are in the language, but "sssmwan" and "sssawmn" are not. How many words are there in the snowman language?

Round 1: Similarity and Pythagorean Th

(1 pt.) $3\sqrt{94}$ (accept only simple radical form)

(2 pts.) $\frac{3}{2}$ or 1.5

(3 pts.) $2\sqrt{13}$ (accept only simple radical form)

Round 2: Algebra I

(1 pt.) 300

(2 pts.) $\frac{2}{3}$ or $0.\overline{6}$ or 0.667

(3 pts.) 10,000

Round 3: Functions

(1 pt.) $\sqrt[3]{x^3-8}$

(2 pts.) 134

(3 pts.) 44

Round 4: Combinatorics

(1 pt.) 120

(2 pts.) 3780

(3 pts.) 384

Round 5: Analytic Geometry

(1 pt.) 17

(2 pts.) $-3 < c < 3$ or $|c| < 3$

(3 pts.) 495

January 28, 2009 TEAM ROUND ANSWERS WOCOMAL Varsity Meet

(2 points each)

1. $6x^2 + 24x + 22$ or $2(3x^2 + 12x + 11)$

2. 4

3. $5\sqrt{13}$ (accept only simple radical form)

4. 3

5. 17

6. $\frac{45}{8}$ or $5\frac{5}{8}$ or 5.625

7. 22

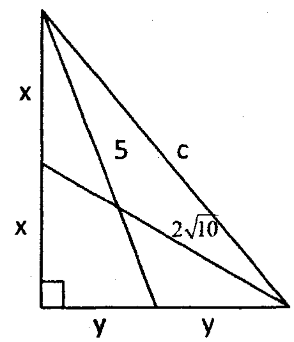
8. \$6.15

9. 217

------------------ Similarity and Pythagorean Theorem --------------------

1. The diagonals of a rhombus divide the rhombus into four congruent right triangles. Let $2x$ be the length of the second diagonal. Use the Pythagorean theorem to get $\left(\frac{15\sqrt{2}}{2}\right)^2 + x^2 = 18^2$. Solving $x = \frac{3\sqrt{94}}{2}$, so $2x = \boxed{3\sqrt{94}}$.

2. $\triangle XYZ \sim \triangle GED$. Then $\frac{DE}{DG} = \frac{YZ}{XZ} = \frac{15}{10} = \boxed{\frac{3}{2} \text{ or } 1.5}$.



3. $c^2 = 4x^2 + 4y^2 = 4(x^2 + y^2)$.
Now $x^2 + 4y^2 = 40$ and $4x^2 + y^2 = 25$.
$\Rightarrow 5x^2 + 5y^2 = 65$ or $x^2 + y^2 = 13$.
$\therefore\ c^2 = 4(13)$ or $c = \boxed{2\sqrt{13}}$.

------------------ Algebra I --------------------

1. Let $x =$ the length of the bridge. Then $x - \frac{x}{2} - \frac{x}{3} = 50$. Solving $x = \boxed{300}$ m.

2. $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{20}{30} = \boxed{\frac{2}{3} \text{ or } 0.\bar{6} \text{ or } 0.667}$.

3. $24{,}999{,}999 - 1 = 5000^2 - 1^2 = (5000+1)(5000-1) = (5001)(4999)$. $50001 + 4999 = \boxed{10{,}000}$.

-------------------- Functions --------------------

1. The inverse of raising to the third power is cube root. So $g(x) = \boxed{\sqrt[3]{x^3 - 8}.}$

2. Let $y = \frac{\sqrt{3x-2}}{4}$. To determine F^{-1}, let $x = \frac{\sqrt{3y-2}}{4}$ and solve for y to get $y = \frac{16x^2 + 2}{3} = F^{-1}(x)$.
$\therefore\ F^{-1}(5) = \frac{16(25)+2}{3} = \frac{402}{3} = \boxed{134}$.

3 Note: $f(x)=\dfrac{(3-\frac{4}{x})^2}{9-\frac{16}{x^2}}=\dfrac{(3-\frac{4}{x})(3-\frac{4}{x})}{(3-\frac{4}{x})(3+\frac{4}{x})}=\dfrac{(3-\frac{4}{x})}{(3+\frac{4}{x})}=\dfrac{3x-4}{3x+4}$. Hence $f(\frac{5}{3}+y)=\dfrac{5+3y-4}{5+3y+4}=\dfrac{3y+1}{3y+9}$

and $f(\frac{5}{3})=\dfrac{5-4}{5+4}=\dfrac{1}{9}$. $\therefore g(y)=\dfrac{1}{y}(\dfrac{3y+1}{3y+9}-\dfrac{1}{9})=\dfrac{1}{y}(\dfrac{27y+9-3y-9}{9(3y+9)})=\dfrac{1}{y}(\dfrac{24y}{3(9y+27)})=\dfrac{8}{9y+27}$.

Hence a=8, b=9, c=27, and a+b+c=$\boxed{44}$.

--------------------- Combinatorics ---------------------

1. ${}_{10}C_7=\boxed{120}$.

2. $({}_9C_1)({}_8C_2)({}_6C_2)=(9)(28)(15)=\boxed{3780}$.

3. Any such path will lay entirely along two faces of the cube. There are 6 such pairs of faces and each yields ${}_9C_3$ paths. But we have to subtract the paths which are associated with more than one pair of faces. These paths traverse a face from corner to corner then follow an edge of the cube, or vice versa. There are also 6 such face-edge pairs; each yielding ${}_6C_3$ paths. Hence $6({}_9C_3)-6({}_6C_3)=6(84)-6(20)=\boxed{384}$ paths.

------------------ Analytic Geometry ------------------

1. $x^2-10x+25+y^2+4y+4=29-k=12\Rightarrow k=\boxed{17}$.

2. To be a hyperbola, $c^2-9<0\Rightarrow c^2<9$. $\therefore \boxed{-3<c<3 \text{ or } |c|<3}$.

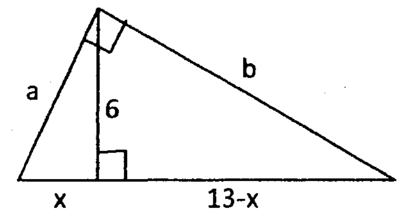
3. Let (x,y) represent all centers. Then $\sqrt{(x-3)^2+(y-3)^2}=\sqrt{(x+6)^2+(y+2)^2}$. Hence $x^2-6x+9+y^2-6y+9=x^2+12x+36+y^2+4y+4\Rightarrow -18x-10y-22=0$ or $9x+5y+11=0$. Thus A=9, B=5, C=11, and $|ABC|=\boxed{495}$.

-------------------- Team --------------------

1. $f(g(x))=3(2x+4)^2+2=3(4x^2+16x+16)+2=12x^2+48x+50$.
$h(f(g(x)))=\frac{1}{2}(12x^2+48x+50)-3=\boxed{6x^2+24x+22 \text{ or } 2(3x^2+12x+11)}$.

2. Since $|x|=6-|x|\Rightarrow x=\pm3$ and $|y|=2\Rightarrow y=\pm2$, there are $\boxed{4}$ ordered pairs. The pairs are $(-3,-2)$, $(-3,2)$, $(3,-2)$, and $(3,2)$.

-------------------- Team (continued) --------------------



3. In a right triangle $\frac{x}{6}=\frac{6}{13-x}$. $\Rightarrow x^2-13x+36=0$ or $x=4,9$.
Then $a^2=4(13)$ and $b^2=9(13)\Rightarrow a=2\sqrt{13}$ and $b=3\sqrt{13}$.
So a+b = $\boxed{5\sqrt{13}}$.

4. Since 5!=120, all factorial greater than 5! have a unit digit of 0. So all have to do is find the unit unit digit of 1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33. (Or just add the unit digits of the four factorials. 1+2+6+4=13.) Therefore the unit digit for the sum of the fifteen factorials is $\boxed{3}$.

5. When $5<C<9 \Rightarrow \frac{x^2}{pos}+\frac{y^2}{neg}=1$ is a hyperbola. So C = almost 9.
When $-\infty<D<8\Rightarrow \frac{x^2}{pos}+\frac{y^2}{pos}=1$ is an ellipse. So D = almost 8.
When 2E+8=4E or E=4 $\Rightarrow \frac{x^2}{16}+\frac{y^2}{16}=1$ is a circle. So E = 4.
$\frac{CD}{E}=\frac{\text{almost } 72}{4}$ = almost 18. So the greatest integer of almost 18 is $\boxed{17}$.

6. $f(1)=2f(2)=4f(3)=8\Rightarrow f(1)=8, f(2)=4$, and f(3)=2.
$$f(4)=f(3+1)=\frac{f(3)-f(2)+10}{f(1)}=\frac{2-4+10}{8}=1.$$
$$f(5)=f(4+1)=\frac{f(4)-f(3)+10}{f(2)}=\frac{1-2+10}{4}=\frac{9}{4}.$$
$$f(6)=f(5+1)=\frac{f(5)-f(4)+10}{f(3)}=\frac{\frac{9}{4}-1+10}{2}=\frac{\frac{9}{4}+\frac{36}{4}}{2}=\boxed{\frac{45}{8}\text{ or }5\frac{5}{8}\text{ or }5.625}.$$

-------------------- Team (continued) --------------------

7. List the possibilities and count

sum of 2 - 2:11, 2:02, 2:20

3 - 2:13, 2:31, 3:03, 3:30, 3:12, 3:21

4 - 2:24, 2:42, 3:14, 3:41

5 - 2:35, 2:53, 3:25, 3:52

6 - 2:46, 3:36

7 - 2:57, 3:47

8 - 3:58 $\Rightarrow$ $\boxed{22}$ minutes

8. List the possibilities and figure the cost for each possibility. Then find the sum.

Let p=penny, n=nickel, d=dime, and q=quarter

3p	.03	p,2n	.11	d,2q	.60
3n	.15	p,2d	.21	2d,q	.45
3d	.30	p,2q	.51	n,d,q	.40
3q	.75	n,2d	.25	p,n,d	.16
2p,n	.07	n,2q	.55	p,n,q	.31
2p,d	.12	2n,d	.20	p,d,q	.36
2p,q	.27	2n,q	.35	Total =	$\boxed{\$6.15}$

9. There has to be at least 1 "s" in the word. So there will be 1 "s" and 3 paired units. This is represented by (1)(3)(3)(3)=27. However, order matters, so the "s" can be placed in ${}_4C_1$ places. $(27)({}_4C_1)=(27)(4)=108$. Also there can be 3, 5, or 7 "s"s. For 3 "s"s $(1)(1)(1)(3)(3)({}_5C_3)=(9)(10)=90$. For 5 "s"s, $(1)(1)(1)(1)(1)(3)({}_6C_5)=(3)(6)=18$. For 7 "s"s, 1. So $108+90+18+1=\boxed{217}$.