December 6, 2000 WOCOMAL Varsity Meet

ROUND I: Definitions

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Define s(n) as the sum of the digits of integer n. Let $s^2(n) = s(s(n))$ and $s^3(n) = s(s(s(n)))$ and so on. What is the value of $s^{1776}(1776)$?

2. A 5-digit positive integer is a *mountain number* if the first three digits are in ascending order and the last three digits are in descending order. For example, 35761 is a mountain number, but 32323 and 33420 are not. How many 5-digit numbers greater than 70000 are mountain numbers?

3. The symbol $\prod_{n=1}^{3}(\exp ression)$ means the product of the values of the expression when n has integral values from 1 to 3. For example, $\prod_{n=1}^{3} 2n = 2(1) \times 2(2) \times 2(3) = 48$.
   If $\prod_{n=1}^{50}\left(\frac{25^n}{5^{101-2n}}\right) = 5^x$, find x.

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. ________________

(3pts) 3. ________________

Algonquin, Bancroft, Burncoat

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ROUND II: Algebra 1 - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Jack paddled 15miles downstream in 2 hours and returned in 3 hours. What was his average rate of speed for the round trip?

2. If $3^x = 5$, find the value of 3^{2x+4}.

3. Find the sum of the integers that satisfy both $3x \le 4 - x^2$ and $x^2 - 1 \ge x + 5$ simultaneously.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3pts) 3. ______________________

Assabet Valley, Quaboag, St..John's, South

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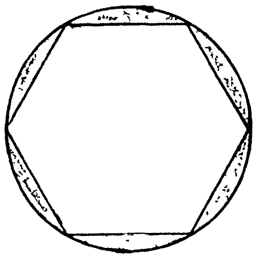
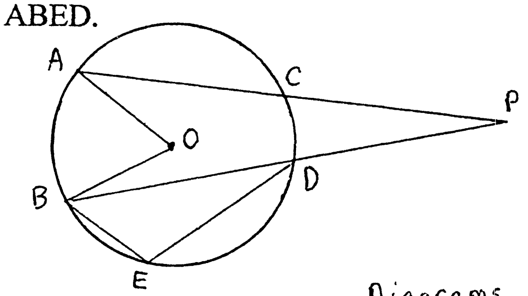
ROUND III: Circles and polygons

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Let a regular hexagon be inscribed in a circle of radius 10. Find the area of the shaded region. In your answer keep π as π and simplify any radicals that occur.

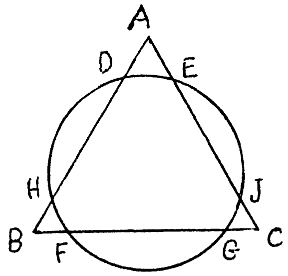


2. Given circle O and degree measures: angle BED = $100°$, arc AC = $4x°$, arc CD = $x°$, angle CPD = $x°$. Find the number of degrees in arc ABED.



Diagrams <u>not</u> to scale

3. Δ ABC is equilateral with sides of length 16. The circle and the triangle intersect in 6 points. HB = 1, DH = 13, EJ = 7. Find JC.



ANSWERS

(1 pt) 1. ______________________________

(2 pts) 2. ______________________________

(3 pts) 3. ______________________________

Bartlett, Hudson

ROUND IV: Sequences and series

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the value of $1 + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

2. What is the least number of aluminum cans needed to build a triangular display 120 cm tall? Each can is 12 cm tall. The base of the triangle is a single row of cans and each can in the rows above it rests on two cans below it , and so on to one can on top.

3. A set of seven books was published at 9-year intervals. When the 7th book was published, the sum of the publication years was 13,601. In what year was the 4th book published?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Bancroft, Doherty, Tantasqua

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ROUND V: Matrices and systems of equations

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM EXCEPT WHEN THE PROBLEM STATEMENT CALLS FOR SOMETHING ELSE

1. Find all (x,y) such that: $\begin{cases} 2\sqrt{x}+4\sqrt{y}=10 \\ 2\sqrt{x}-3\sqrt{y}=3 \end{cases}$

2. Multiplying $\begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ by $\begin{bmatrix} .94 & -.34 \\ .34 & .94 \end{bmatrix}$ rotates the triangle with vertices (2,2); (1,4); (0,0) 20 degrees counterclockwise about the origin. One of the vertices of the new triangle is in the second quadrant. Give its coordinates to 2 decimal places.

3. Solve this system for all ordered pairs (x,y): $\begin{cases} x^2 - xy = 28 \\ y^2 - xy = 21 \end{cases}$

ANSWERS

(1 pt) 1. ______________________________

(2 pts) 2. (,)

(3 pts) 3. ____

Algonquin, Bromfield, Worcester Academy

TEAM ROUND: Topics of previous rounds and open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM EXCEPT WHEN THE PROBLEM STATEMENT CALLS FOR SOMETHING ELSE 2 points each

1. Let $D = a^2 + b^2 + c^2$ where a and b are consecutive integers and c = ab. Then $\sqrt{D}$ is:
   a) always an even integer
   b) sometimes an odd integer, sometimes not
   c) always an odd integer
   d) sometimes a rational number, sometimes not
   e) none of the above

2. The sum of two numbers is 50 and the positive difference of their reciprocals is $\frac{3}{40}$. 10 and 40 are two such numbers. There is another pair also, one positive and one negative. Find the positive one.

3. A circle is inscribed in a regular pentagon of perimeter 32.5 inches. Find the radius of the circle to the nearest tenth.

4. If this pattern is continued, what will be the last number in the 40th row?

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

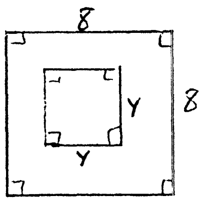
5. Find all the ordered triples of real numbers (x,y,z) which satisfy:

$$(x + y)(x + y + z) = 120$$
$$(y + z)(x + y + z) = 96$$
$$(x + z)(x + y + z) = 72$$

6. Find the sum of the x-intercepts of the following:

$$y = |x - 4| \quad , \quad |x| + |y| = 10 \quad , \quad y = |x| - 5$$

7. Darts thrown at a board are equally likely to hit anywhere within the region on the board. If 75% of the darts hit inside the small square, what is the value of y to the nearest tenth?



8. Let a, b and c be positive integers whose square roots are the lengths of the sides of a right triangle. Find the least possible value of the sum a+b+c.

9. If 6 boys can build 6 houses in 6 days and 12 girls can build 12 houses in 12 days, how many houses can 12 boys and 12 girls build in 12 days?

Auburn, Bartlett, Burncoat, Hudson, North, St. John's, Tahanto, unspecified

December 6, 2000 WOCOMAL Varsity Meet ANSWERS

ROUND I
defs
1. 1 pt 3
2. 2 pts 36
3. 3 pts 50

ROUND II
alg 1
1. 1 pt 6 miles/hr (mph OK) need units
2. 2 pts 2025
3. 3 pts −9

ROUND III
circles polygons
1. 1 pt $100\pi - 150\sqrt{3}$ factored OK
2. 2 pts 235° Just 235 OK
3. 3 pts 6

ROUND IV
seq series
1. 1 pt 3
2. 2 pts 55
3. 3 pts 1943

ROUND V
mat. systems
1. 1 pt (9,1) or $x=9$, $y=1$
2. 2 pts (−.42, 4.10)
3. 3 pts (4,−3), (−4,3) need both

TEAM ROUND 2 pts each

1. C
2. $\frac{200}{3}$ or $66\frac{2}{3}$ or $66.\bar{6}$
3. 4.5 in
4. 1680
5. (4,6,2), (−4,−6,−2) need both
6. 4
7. 6.9
8. 4
9. 36

December 6, 2000 WOCOMAL VARSITY MEET BRIEF SOLUTIONS

ROUND I

1. $s(1776) = 21$
$s(21) = 3$
$s(3) = 3$, etc so $s^{1776}(1776) = 3$

2. We must start 789 _ _.
For 7898 $\underline{x}$ there are 8 possible x's
7897 $\underline{x}$ 7
⋮ ⋮
7891 $\underline{x}$ 1
$8+7+6+5+4+3+2+1 = 36$

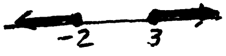
3. Write the product as
$$\prod_{n=1}^{50} \frac{5^{2n}}{5^{101-2n}} = \prod_{n=1}^{50} 5^{4n-101}$$
$$= 5^{-97-93-89-\cdots+91+95+99}$$
The exponent is a 50 term arith. prog
with sum $\left(\frac{-97+99}{2}\right)(50) = 50$
So $5^{50} = 5^x$ and $x = 50$

ROUND II

1. Ave speed $= \frac{\text{total distance}}{\text{total time}} = \frac{30 mi}{5 hr} = 6 \frac{mi}{hr}$

2. $3^{2x+4} = (3^x)^2 \cdot 3^4 = 5^2 \cdot 81 = 2025$

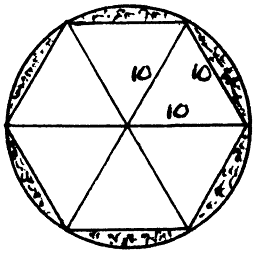
3. $x^2+3x-4 \le 0$ AND $x^2-x-6 \ge 0$
$(x+4)(x-1) \le 0$ $(x-3)(x+2) \ge 0$



Integers in intersection are $-4, -3, -2$.
Their sum is -9.

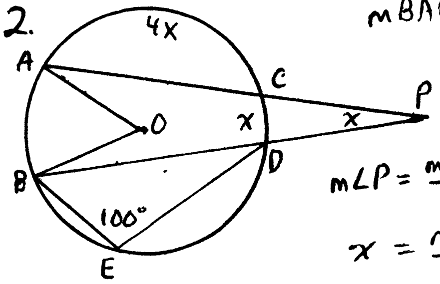
ROUND III

1. Circle area $= 100\pi$
hexagon area, 6 equilat Δ's with side 10, is
$6 \cdot \frac{10^2}{4}\sqrt{3} = 150\sqrt{3}$
Ans. $100\pi - 150\sqrt{3}$



ROUND III cont.

2.



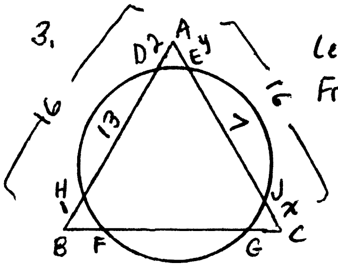
$m\overset{\frown}{BACD} = 200°$
from $\angle BED$

$m\angle P = \frac{m\overset{\frown}{AB} - m\overset{\frown}{CD}}{2}$

$x = \frac{m\overset{\frown}{AB} - x}{2} \Rightarrow m\overset{\frown}{AB} = 3x$

$m\overset{\frown}{BACD} = 3x + 4x + x = 200 \Rightarrow x = 25,\ m\overset{\frown}{AB} = 75°$
$m\overset{\frown}{BED} = 360 - 200 = 160°$
$m\overset{\frown}{ABED} = m\overset{\frown}{AB} + m\overset{\frown}{BED} = 75 + 160 = 235°$

3.



Let $AE = y$.
From thm about secants,
$15(2) = (7+y)y$
$0 = y^2 + 7y - 30$
$0 = (y+10)(y-3)$
$y = 3$
$x = 16 - (3+7) = 6$

ROUND IV

1. 1 + infinite geom series with $a = \frac{2}{3}$, $r = \frac{2}{3}$
$1 + \frac{\frac{2}{3}}{1-\frac{2}{3}} = 1 + 2 = 3$

2. Must be $\frac{120}{12} = 10$ cans tall.
One can on top, then $2, 3, \cdots, 10$
$1+2+3+\cdots+10 = \left(\frac{1+10}{2}\right)(10) = 55$

3 Let x = year 4th book published.
$(x-27) + (x-18) + (x-9) + x + (x+9) + (x+18) + (x+27) = 13601$
$7x = 13601$ and $x = 1943$

ROUND V

1. subtract to get $7\sqrt{y} = 7$, so $y = 1$
Then $2\sqrt{x} + 4 = 10$ gets $\sqrt{x} = 3$ and $x = 9$
$(x, y) = (9, 1)$