Worcester County Mathematics League

Freshman Meet 3 – March 2, 2005

Round 1: Graphing on a Number Line

NO CALCULATOR ALLOWED

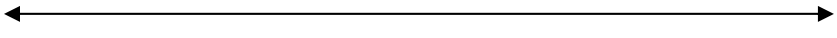
Draw the graph of each of the following problems on the corresponding number line provided below. Please specify all endpoints on your graph.

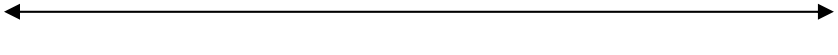
1. $\frac{x-2}{2} > \frac{4x-3}{4}$

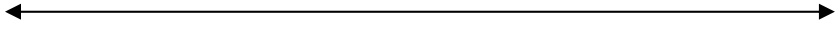
2. $(x-3)(x-1) > 0$

3. $|x+1| \le x+1$

ANSWERS

(1 pt.) 1. 

(2 pts.) 2. 

(3 pts.) 3. 

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Round 2: Operations on Polynomials

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

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1. After the expansion and simplification of $(y-3)(y^3-2y+2)$, find the coefficient of the polynomial's linear term.

2. If $P=4x+3$ and $Q=3x+4$, find P^2-Q^2 as a single polynomial. DO NOT FACTOR YOUR ANSWER.

3. Factor $(x-2)(x^2-1)-6x-6$ as the product of three (not necessarily distinct) binomials.

ANSWERS

(1 pt.) 1._____

(2 pts.) 2._____

(3 pts.) 3._____

Quaboag, Worc. Academy (2, 3)

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Round 3: Techniques of Counting and Probability

All answers must be in simplest exact form

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1. A student may choose to complete any 8 out of the possible 12 questions on an exam. How many different ways can the student complete the exam?

2. There are seven teams in a soccer league. In a season, each team plays each of the other teams exactly once. How many different games need to be scheduled in a single season?

3. An urn contains 5 white and 7 red balls. If 4 balls are randomly drawn from the urn without replacement, what is the probability that they will all be red? Express your answer as a reduced fraction.

ANSWERS

(1 pt.) 1._____

(2 pts.) 2._____

(3 pts.) 3._____

Doherty, Bancroft, South

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Round 4: Perimeter, Area and Volume

All answers must be in simplest exact form

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1. Two rectangular solids have equal volumes. If their respective dimensions are 2, 4, x and 3, 6, x –5, find the value of x.

2. A rectangle has a perimeter of 2 yards and a length of 22 inches. Find the *area* of the rectangle in *square inches*.

3. Find the total surface area of a 3 by 3 by 3 cube through which a 1 by 1 by 3 square hole has been punched. Assume that the edges of the hole are parallel to the edges of the cube.

ANSWERS

(1 pt.) 1._____

(2 pts.) 2._____ Square inches

(3 pts.) 3._____

Southbridge, Worc. Academy (2, 3)

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TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places! (3 pts. each)

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1. If $3x + 2 = 8$, find the value of $\frac{1}{3}x + \frac{1}{2}$.

2. If each side of a square is increased by 3 feet, the area is increased by 33 square feet. Find the length of a side of the original square, in feet.

3. Find the units' digit of $3^{2005} - 2^{2005}$.

4. Expand and simplify $(x-1)^3 - (x-1)^2 - (x-1)$ as a single polynomial. DO NOT FACTOR YOUR ANSWER.

5. Eight hundred and fifty-two digits are used to number the pages of a book consecutively beginning with page 1. Find the number of pages in the book.

6. On the space provided on the answer sheet, graph the solution set of:

$$x + 2 < -x < 5$$

7. Kyle, Lou and Mike all run on Uxbridge's cross-country team. At their first meet, Kyle did the best of the three, finishing exactly in the middle among all participants. Lou finished in 14^{th} place overall, and Mike was 24^{th}. How many runners participated in the first meet?

8. A set of digits consists of one 1, two 9s and one 7. A pair of these digits is randomly selected without replacement. Find the probability that the pair can be arranged to form a two-digit prime number.

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All answers must be in simplest exact form!

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ANSWER SHEET – TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places! (3 pts. each)

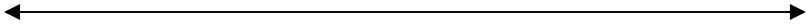
1. _____

2. _____ feet

3. _____

4. _____

5. _____ pages

6. 

7. _____

8. _____

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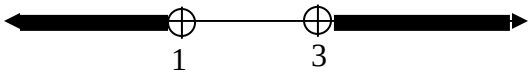
ANSWERS

Round 1

1. 

2. 

3. 

Round 2

1. 8

2. $7x^2 - 7$

3. $(x-4)(x+1)(x+1) = (x-4)(x+1)^2$

Round 3

1. 495

2. 21

3. $\frac{7}{99}$

Round 4

1. 9

2. 308

3. 64

Team Round

1. $\frac{7}{6} = 1\frac{1}{6} = 1.1\overline{6} \approx 1.167$

2. 4

3. 1

4. $x^3 - 4x^2 + 4x - 1$

5. 320

6. 

7. 25

8. $\frac{5}{6} = 0.8\overline{3} \approx 0.833$

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BRIEF SOLUTIONS

Round 1

1. $\frac{x-2}{2} > \frac{4x-3}{4} \Rightarrow 2x-4 > 4x-3 \Rightarrow x < -\frac{1}{2}$

2. $(x-3)(x-1) > 0 \Rightarrow x > 3$ or $x < 1$

3. $|x+1| \le x+1 \Rightarrow x+1 \ge -x-1$ and $x+1 \le x+1$. Then,
$x+1 \ge -x-1 \Rightarrow 2x \ge -2 \Rightarrow x \ge -1$

Round 2

1. $(y-3)(y^3-2y+2) = y^3-5y^2+8y-6 \Rightarrow$ the coefficient of the linear term is 8.

2. $P^2 = (4x+3)^2 = 16x^2+24x+9$ and $Q^2 = (3x+4)^2 = 9x^2+24x+16$. Hence,
$P^2 - Q^2 = 7x^2 - 7$.

3. $(x-2)(x^2-1)-6x-6 = (x-2)(x+1)(x-1)-6(x+1) = (x+1)\left[(x-2)(x-1)-6\right]$
$= (x+1)(x^2-3x-4) = (x+1)(x+1)(x-4)$.

Round 3

1. The number of possible combinations is ${}_8C_{12} = \frac{12!}{4!8!} = 495$.

2. Each team plays 6 games. Therefore all teams play 42 games. But, in this case, since each game is counted twice, 21 games need to be scheduled.

3. The probability of the first ball being red is $\frac{7}{12}$. The probability of the second ball being red is $\frac{6}{11}$. The probability of the first ball being red is $\frac{5}{10}$. The probability of the first ball being red is $\frac{4}{9}$. Therefore, the probability of selecting all red balls is $\frac{7}{12} \cdot \frac{6}{11} \cdot \frac{5}{10} \cdot \frac{4}{9} = \frac{7}{99}$.

Round 4

1. If the volumes are equal, then $8x = 18(x-5) \Rightarrow x = 9$.

2. The perimeter of the rectangle is 72 inches. If the length is 22 inches, then the width is $\frac{1}{2}(72-44) = 14$ inches. Hence, the area of the rectangle is $14 \cdot 22 = 408$ sq. inches.

3. The surface area of the "un-punched" cube is $6 \cdot 3^2 = 54$. Hence, the surface area of the outside of the "punched" cube is 52. Also, the area of the inside surface of the hole is $4 \cdot 3 = 12$. Therefore, the total surface area is 52 + 12 =64.

Team Round

1. $3x + 2 = 8 \Rightarrow 3x = 6 \Rightarrow \frac{1}{3}x = \frac{2}{3} \Rightarrow \frac{1}{3}x + \frac{1}{2} = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$

2. Let x be the length of a side of the square. Then, $(x+3)(x+3) = x^2 + 33$
$\Rightarrow x^2 + 6x + 9 = x^2 + 33 \Rightarrow x = 4$.

3. Explore powers of 2 and 3. The units' digits of the powers of 2 cycle with length 5 (i.e., 2, 4, 8, 6, 2, 4, 8,…). The units' digits of the powers of 3 cycle with length 4 (i.e., 3, 9, 7, 1, 3, 9, 7,…). Since, $2005 \div 5$ leaves no remainder, 2^{2005} ends in a 2. Also, since $2005 \div 4$ leaves a remainder of 1, 3^{2005} ends in a 3. Hence, $3^{2005} - 2^{2005}$ ends in a 1.

4. $(x-1)^3 - (x-1)^2 - (x-1) = x^3 - 3x^2 + 3x + 1 - (x^2 - 2x + 1) - x + 1 = x^3 - 4x^2 + 4x - 1$

5. Pages 1, 2, …, 9 use 9 digits total. Pages 10, 11, 12, …, 99 use 180 total digits. This leaves a total of 852 – 189 = 663 digits or an additional 221 pages. Hence, the total number of pages is 99 + 221 = 320.

6. $x + 2 < -x < 5 \Rightarrow x > -5$ and $2x < -2 \Rightarrow x < -1$.

7. If Mike finished 24th and Kyle finished exactly in the middle, then there are at least 25 runners, with Kyle finishing at least 13th. But, since Lou finished 14th, Kyle couldn't have finished below 13th. So, there were 25 runners.

8. There are ${}_2C_4 = 6$ ways to choose two of the digits. Without regard to the order of the digits they are 19, 19, 17, 79, 79, and 99. From those possibilities the primes that can be formed are: 19, 17, 71, 79 and 97. Hence, the probability is $\frac{5}{6}$.