46th ANNUAL

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**THE MASSACHUSETTS ASSOCIATION
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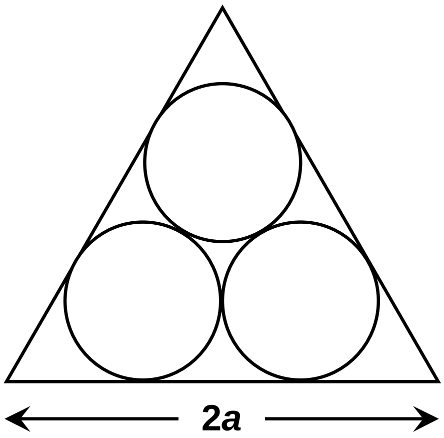
SECOND-LEVEL EXAMINATION

Tuesday, March 2, 2010

SOLUTIONS

1. SNUG CIRCLES

a) Suppose that three circles of the same size are mutually tangent and snugly fit into an equilateral triangle of side $2a$, as shown. What is the area covered by the three circles? Show the work that produced your answer. (7 points)

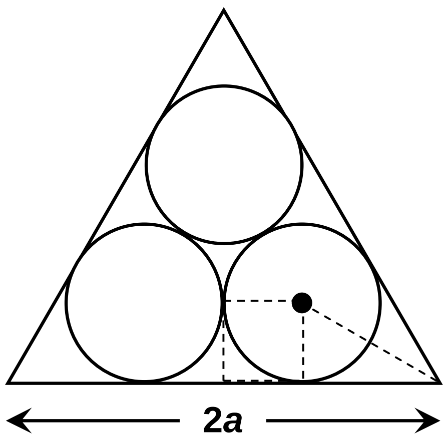


b) Now find three noncongruent, nonoverlapping circles that cover more of the area inside the same triangle. Verify that the covered area is indeed greater than the covered area in part (a). (7 points)

Solution

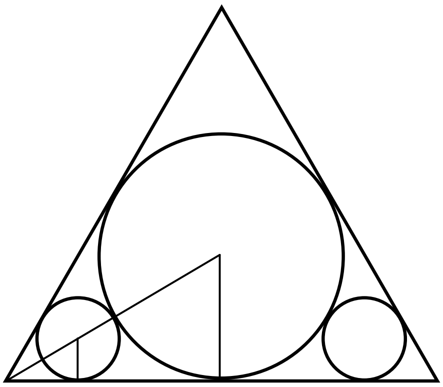
a) First, we note that the area of an equilateral triangle of side length s is $\frac{s^2\sqrt{3}}{4}$. For a triangle of side length $2a$, the area is $a^2\sqrt{3}$.

Next, we take note of the dashes in the figure below. If the radius of the circle is r, then the length of each side of the dashed square is also r, and the base of the 30-60-90 triangle next to it is $a-r$. Because $a - r = r\sqrt{3}$, we see that $r = \frac{a}{\sqrt{3}+1}$, and the area of each circle is therefore $A = \pi r^2 = \pi \frac{a^2}{\left(\sqrt{3}+1\right)^2} = \frac{\pi a^2}{4+2\sqrt{3}}$.



The ratio of the areas of the three circles to the area of the triangle is therefore given by
$$\frac{\dfrac{3\pi a^2}{4+2\sqrt{3}}}{a^2\sqrt{3}}=\frac{3\pi}{\left(4+2\sqrt{3}\right)\sqrt{3}}=\frac{3\pi}{4\sqrt{3}+6}\approx 0.729.$$

b) For three circles to cover more of the triangle's area, it stands to reason that one of them must be the largest circle we can inscribe within the triangle, namely, the incircle. This is shown, along with two other, smaller circles, in the figure.



The radius r_1 of the incircle is one-third the height of the large equilateral triangle: $r_1=\dfrac{a\sqrt{3}}{3}$. The radius r_2 of one of the smaller circles is determined by similar triangles.

Because these are 30-60-90 triangles, the hypotenuse of each triangle is twice the short leg of the triangle, and therefore twice the radius of the related circle. The hypotenuse of the smaller triangle is therefore given by $2r_1-\left(r_1+r_2\right)=r_1-r_2=\dfrac{a\sqrt{3}}{3}-r_2$. Since this is twice the smaller radius, we have $\dfrac{a\sqrt{3}}{3}-r_2=2r_2\Rightarrow\dfrac{a\sqrt{3}}{3}=3r_2\Rightarrow r_2=\dfrac{a\sqrt{3}}{9}$.

The three circles thus have a combined area given by
$\pi r_1^2+2\pi r_2^2=\pi\left[\left(\dfrac{a\sqrt{3}}{3}\right)^2+2\left(\dfrac{a\sqrt{3}}{9}\right)^2\right]=\dfrac{11\pi a^2}{27}$. The ratio of these combined areas to the area of the equilateral triangle is about 0.739, which satisfies the requirement of the problem.

2. LOGS? ABSOLUTELY!

Show that for every $a > 0$ and $b > 0$ (with the further proviso that neither a nor b can equal 1), $\left|\log_a(b)+\log_b(a)\right| \geq 2$. (11 points)

Solution

Let $x=\log_a(b)$. Then, $a^x=b \Rightarrow b^{1/x}=a \Rightarrow \frac{1}{x}=\log_b(a)$. Thus, the problem requires that we show that $\left|x+\frac{1}{x}\right|-2\geq 0$ for all nonzero x. Since x and $\frac{1}{x}$ have the same sign, we have $\left|x+\frac{1}{x}\right|-2=|x|-2+\left|\frac{1}{x}\right|=\left(\sqrt{|x|}-\sqrt{\frac{1}{x}}\right)^2\geq 0$, which completes the proof.

3. I KNOW THE ANSWER, WHAT WAS THE QUESTION?

On a multiple-choice test one of the questions was illegible. The choice of answers was:

a) All of the below

b) None of the below

c) All of the above

d) Exactly one of the above

e) None of the above

f) None of the above

What was the correct answer? Justify your response. (12 points)

Solution

1. If choice (a) is correct, then choice (e) is correct. But choice (e) requires that choice (a) is *incorrect*. The contradiction renders choice (a) incorrect.
2. If choice (b) is correct, then choices (c) and (d) must be incorrect. But if choice (c) is incorrect, then choice (a) is incorrect (since our supposition here is that choice (b) is correct). If choice (a) is incorrect while (b) is correct and (c) is incorrect, then choice (d), which requires one correct answer among the first three, is correct. The contradiction renders choice (b) incorrect.
3. If choice (c) is correct, then choice (a) is correct. But according to scenario 1, this is impossible, which renders choice (c) incorrect.
4. If choice (d) is correct, then one of (a), (b), and (c) is correct. But scenarios 1, 2, and 3 have rendered all of them incorrect. This in turn renders choice (d) incorrect.
5. If choice (f) is correct, then choice (e) is incorrect. But an incorrect (e) requires that one of (a), (b), (c), and (d) must be correct. Since scenarios 1 – 4 have made that impossible, then choice (f) must be incorrect. The contradiction renders choice (f) incorrect.
6. The remaining choice is (e). Is it correct? It is if (a), (b), (c), and (d) are incorrect. Since scenarios 1 – 4 satisfy that condition, then choice (e) is correct.

Finally, since a multiple-choice test requires only one answer to be correct, a correct (e) means that (f) is incorrect, which was demonstrated in scenario 5.

4. A FUNCTIONAL INEQUALITY

Let f and g be any real-valued functions defined on [0,1]. Prove that there exist a and b in [0,1] such that $\left|f(a)+g(b)-ab\right| \geq \frac{1}{4}$. (12 points)

Solution

Suppose that the result is false. That is, suppose that for every a and b in [0,1], $\left|f(a)+g(b)-ab\right| < \frac{1}{4}$.

For a = 0 and b = 1, we have $-\frac{1}{4} < f(0)+g(1) < \frac{1}{4}$, and for a = 1 and b = 0, we have $-\frac{1}{4} < f(1)+g(0) < \frac{1}{4}$. Adding these two inequalities, we have $-\frac{1}{2} < f(0)+f(1)+g(0)+g(1) < \frac{1}{2}$. Call this **Result 1**.

For a = 0 and b = 0, we have $-\frac{1}{4} < f(0)+g(0) < \frac{1}{4}$, and for a = 1 and b = 1, we have $\frac{3}{4} < f(1)+g(1) < \frac{5}{4}$. Adding these two inequalities we have $\frac{1}{2} < f(0)+f(1)+g(0)+g(1) < \frac{3}{2}$. Call this **Result 2**.

Results 1 and 2 are contradictory, meaning that our supposition (namely, that the proposed result is false) cannot be correct. This means, in turn, that the proposed result is true.

5. A RATIO OF AREAS

Given: triangle *AOB* with *A* (1, 2), *B* (4, 0), and *O* (0, 0). Think of *A* and *B* as $A_1(a_1, b_1)$ and $B_1(c_1, d_1)$, respectively, and define the coordinates of A_2 and B_2 as $(a_1 + b_1, a_1 - b_1)$ and $(c_1 + d_1, c_1 - d_1)$, respectively. Further, define the coordinates of A_{n+1} and B_{n+1} as $(a_n + b_n, a_n - b_n)$ and $(c_n + d_n, c_n - d_n)$, respectively.

a) Find the ratio of the area of triangle A_9OB_9 to the area of triangle A_4OB_4. (4 points)

b) For $n = 2k+1$, find a formula for A_{2k+1}. (2 points)

c) For $n = 2k$, find, **with proof**, a formula for A_{2k}. (4 points)

d) For $n = 2k+1$, find a formula for B_{2k+1}. (2 points)

e) For $n = 2k$, find a formula for B_{2k}. (2 points)

f) Find, in terms of *n*, a formula for the area of the triangle with coordinates $O(0,0)$, A_n, and B_n. Prove your result using the formulas from (a) – (d). (4 points)

Solution

Under the transformation $A_{n+1} = (a_n + b_n, a_n - b_n)$, we have $A_1 = (1,2)$, $A_2 = (3,-1)$, $A_3 = (2,4)$, $A_4 = (6,-2)$, $A_5 = (4,8)$, …, $A_9 = (16,32)$. Under the transformation $B_{n+1} = (c_n + d_n, c_n - d_n)$, we have $B_1 = (4,0)$, $B_2 = (4,4)$, $B_3 = (8,0)$, $B_4 = (8,8)$, $B_5 = (16,0)$, …, $B_9 = (64,0)$.

a) Using the coordinates above, we find that triangle A_4OB_4 has an area of 32, while triangle A_9OB_9 has an area of 1024; the ratio is 32.

Transformation A_{n+1} includes two subsequences, relating to subquestions (b) and (c), respectively.

b) For $n = 2k+1$, $A_{2k+1} = (2^k, 2^{k+1})$.

c) For $n = 2k$, $A_{2k} = (3 \cdot 2^{k-1}, -1 \cdot 2^{k-1})$. We prove this by induction:

Base case: For *n* = 2, we have *k* = 1, so $A_2 = (3 \cdot 2^{1-1}, (-1) \cdot 2^{1-1}) = (3,-1)$.

Inductive step: Assume that $A_{2k} = \left(3 \cdot 2^{k-1}, -1 \cdot 2^{k-1}\right)$. Then,
$A_{2k+1} = \left(3 \cdot 2^{k-1} + (-1)2^{k-1}, 3 \cdot 2^{k-1} - (-1) \cdot 2^{k-1}\right) = \left(2 \cdot 2^{k-1}, 4 \cdot 2^{k-1}\right) = \left(2^k, 2^{k+1}\right)$. Then,
$A_{2k+2} = \left(2^k + 2^{k+1}, 2^k - 2^{k+1}\right) = \left(2k + 2 \cdot 2^k, 2k - 2 \cdot 2^k\right) = \left(3 \cdot 2^k, (-1)2^k\right)$, and the formula is proved.

Transformation B_{n+1} also includes two subsequences, relating to subquestions (d) and (e), respectively.

d) For $n = 2k+1$, $B_{2k+1} = \left(2^{k+2}, 0\right)$.

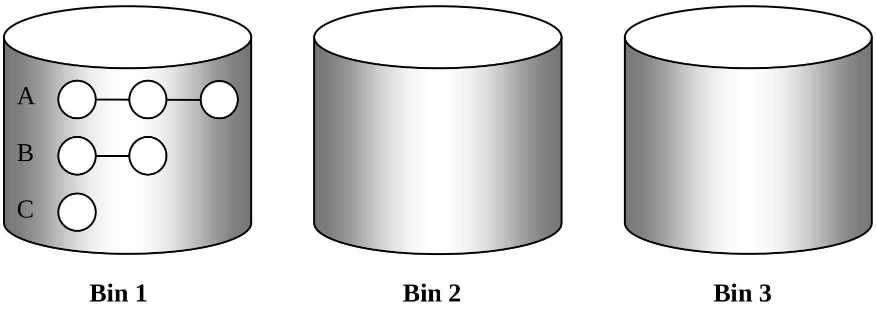
e) For $n = 2k$, $B_{2k} = \left(2^{k+1}, 2^{k+1}\right)$.

f) We now use all the results from (b)–(e). For $n = 2k+1$, the points are $O(0,0)$, $A_{2k+1} = \left(2^k, 2^{k+1}\right)$, and $B_{2k+1} = \left(2^{k+2}, 0\right)$. Thus, the area is $\frac{1}{2} \cdot 2^{k+1} \cdot 2^{k+2} = 2^{(2k+1)+1} = 2^{n+1}$.
For n = 2k, the points are $O(0,0)$, $A_{2k} = \left(3 \cdot 2^{k-1}, (-1)2^{k-1}\right)$, and $B_{2k} = \left(2^{k+1}, 2^{k+1}\right)$.
Using determinants, the area is
$\frac{1}{2}\begin{vmatrix} (3)2^{k-1} & (-1)2^{k-1} \\ 2^{k+1} & 2^{k+1} \end{vmatrix} = \frac{1}{2}\left[(3)2^{2k} - (-1)2^{2k}\right] = \frac{1}{2} \cdot 4 \cdot 2^{2k} = 2^{2k+1} = 2^{n+1}$. Thus, the sequence of areas is a geometric sequence with common ratio 2, and in terms of n, the area of the nth triangle is 2^{n+1}.

6. BIN THERE, DONE THAT

Sam has three bins, Bin 1, Bin 2, and Bin 3, the first of which contains three strings of balls, String A, String B, and String C. He selects a string at random by reaching into Bin 1, grabbing a ball, and pulling out that ball and the entire string connected to it. Each ball has an equal probability of being selected, but each string does not. Sam then places this string of balls into Bin 2. In the same manner, he selects at random one of the two strings remaining in Bin 1 and places that one into Bin 2 as well. Finally, again in the same manner of selection, he selects at random one of the two strings of balls in Bin 2 and places it into Bin 3.

a) Let String A consist of 3 balls, String B consist of 2 balls, and String C consist of 1 ball, as depicted below. When following the selection process described above, what is the probability that String A is placed into Bin 3? (5 points)



b) If String A consists of a balls, String B consists of b balls, and String C consists of c balls, what is the probability, in terms of a, b, and c, that String A is placed into Bin 3? (5 points)

c) Suppose that the numbers of balls on the strings form an arithmetic series, such that String A consists of a balls, String B consists of $a + d$ balls, and String C consists of $a + 2d$ balls. Find, in simplified form, the probability that String B is placed into Bin 3. (5 points)

Solution

(a) Let $P(a,b)$ denote the probability of choosing strings of length *a* and *b* from Bin 1. Let $P_b(a)$ denote the probability of choosing the string of length *a* from Bin 2 when strings of length *a* and *b* are in Bin 2. Note that $P_b(a) = \frac{a}{a+b}$ and that

$$P(a,b) = \frac{a}{a+b+c} \cdot \frac{b}{b+c} + \frac{b}{a+b+c} \cdot \frac{a}{a+c} = \frac{a \cdot b}{a+b+c} \cdot \left(\frac{1}{b+c} + \frac{1}{a+c} \right).$$

In general, the probability that String A is placed in Bin 3 is given by
$P(a,b) \cdot P_b(a) + P(a,c) \cdot P_c(a)$, which, building from the above result, is equal to

$$\frac{a \cdot b}{a+b+c} \cdot \left(\frac{1}{b+c} + \frac{1}{a+c} \right) \cdot \frac{a}{a+b} + \frac{a \cdot c}{a+b+c} \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} \right) \cdot \frac{a}{a+c}.$$

When *a* = 3, *b* = 2, and *c* = 1, we have

$$\frac{3 \cdot 2}{3+2+1} \cdot \left(\frac{1}{2+1} + \frac{1}{3+1} \right) \cdot \frac{3}{3+2} + \frac{3 \cdot 1}{3+2+1} \cdot \left(\frac{1}{3+2} + \frac{1}{2+1} \right) \cdot \frac{3}{3+1}$$

$$= \frac{6}{6} \cdot \left(\frac{1}{3} + \frac{1}{4} \right) \cdot \frac{3}{5} + \frac{3}{6} \left(\frac{1}{5} + \frac{1}{3} \right) \cdot \frac{3}{4} = \frac{126}{360} + \frac{72}{360} = \mathbf{\frac{11}{20}}$$

b) The probability of choosing the string of length *a* from Bin 2 is, as determined above, $P(a,b)\cdot P_b(a)+P(a,c)\cdot P_c(a)$. In the general case, therefore,

$$P(a,b)\cdot P_b(a)+P(a,c)\cdot P_c(a)$$

$$=\frac{a\cdot b}{a+b+c}\cdot\left(\frac{1}{b+c}+\frac{1}{a+c}\right)\cdot\frac{a}{a+b}+\frac{a\cdot c}{a+b+c}\cdot\left(\frac{1}{a+b}+\frac{1}{b+c}\right)\cdot\frac{a}{a+c}$$

$$=\frac{a^2b\cdot(a+b+2c)}{(a+b+c)(a+b)(b+c)(c+a)}+\frac{a^2c\cdot(a+2b+c)}{(a+b+c)(a+b)(b+c)(c+a)}$$

$$=\frac{a^2\left[b(a+b+2c)+c(a+2b+c)\right]}{(a+b+c)(a+b)(b+c)(c+a)}=\frac{a^2\left[b(a+b+c)+bc+c(a+b+c)+bc\right]}{(a+b+c)(a+b)(b+c)(c+a)}$$

$$=\frac{a^2\left[(a+b+c)(b+c)+2bc\right]}{(a+b+c)(a+b)(b+c)(c+a)}$$

We see, by substitution, that if *a* = 3, *b* = 2, and *c* = 1, we get from this the same result of $\frac{11}{20}$ as we got in (a).

c) Switching *a* and *b* in the formula derived in (b) gives the probability of placing String B into Bin 3: $P(b,a)\cdot P_a(b)+P(b,c)\cdot P_c(b)=\dfrac{b^2\left[(a+b+c)(a+c)+2ac\right]}{(a+b+c)(a+b)(b+c)(c+a)}$.

If *a* = *a*, *b* = *a* + *d*, and *c* = *a* + 2*d*, we substitute these values into the above expression:

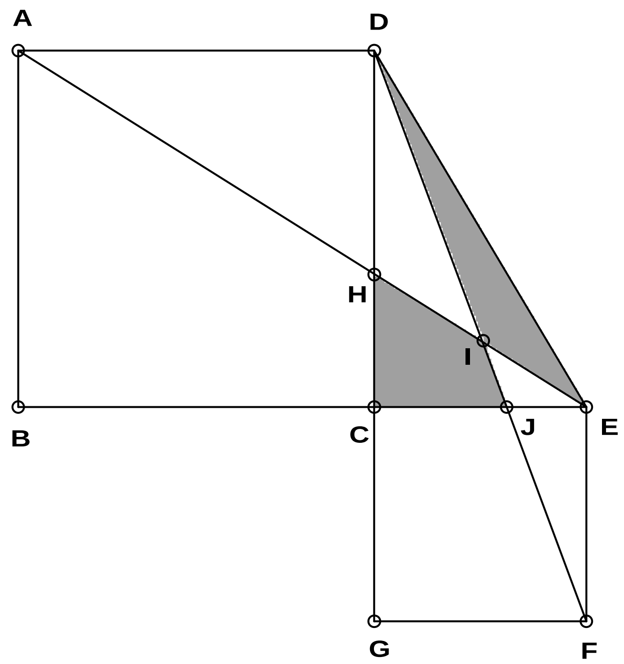
$$\frac{(a+d)^2\left[(3a+3d)(2a+2d)+2a(a+2d)\right]}{(3a+3d)(2a+d)(2a+3d)(2a+2d)}=\frac{(a+d)^2\left[6(a+d)^2+2a(a+2d)\right]}{6(a+d)^2(2a+d)(2a+3d)}$$

$$=\frac{3(a+d)^2+a(a+2d)}{3(2a+d)(2a+3d)}=\frac{3a^2+6ad+3d^2+a^2+2ad}{3\left(4a^2+8ad+3d^2\right)}=\frac{4a^2+8ad+3d^2}{3\left(4a^2+8ad+3d^2\right)}=\frac{1}{3}$$

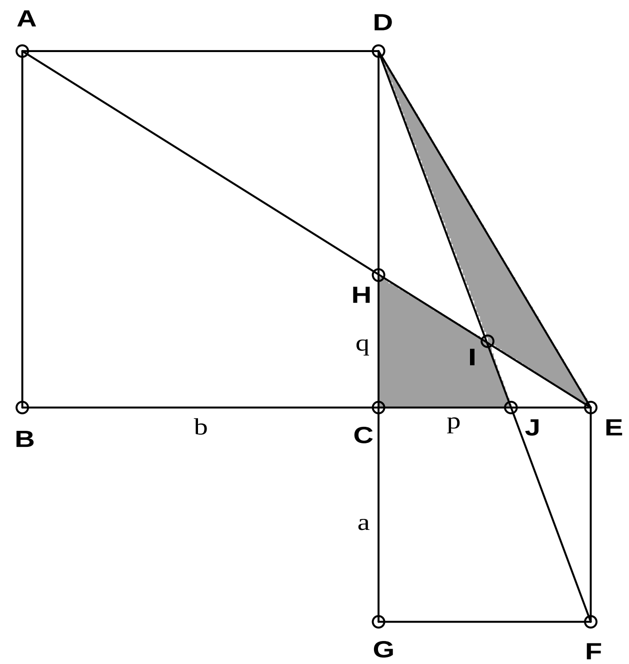
The probability of placing String B into Bin 3 is $\frac{1}{3}$, regardless of the values of *a* and *d*.

7. THE SAME SHADE

In the figure, ΔCDE is a right triangle, $ABCD$ and $CEFG$ are squares, and I is the intersection of $\overline{AE}$ and $\overline{DF}$. Show that the shaded regions in the triangle have the same area. (18 points)



Solution



Let a be the side length of the smaller square, and let b be the side length of the larger square. Let p be the length of $\overline{CJ}$, and let q be the length of $\overline{CH}$. Triangles DCJ and DGF are similar, as are triangles EHC and EAB. Therefore:

$\frac{p}{a}=\frac{b}{a+b} \Rightarrow p=\frac{ab}{a+b}$ and $\frac{q}{b}=\frac{a}{a+b} \Rightarrow q=\frac{ab}{a+b}$, so that $p = q$.

Since the length of $\overline{JE}$ is $a - p$, the area of triangle DJE is given by $\frac{1}{2}b(a-p)$, and the area of triangle DIE is given by $\frac{1}{2}b(a-p)-M$, where M is the area of triangle IJE.

Since the area of triangle EHC is given by $\frac{1}{2}aq$, the area of quadrilateral $CHIJ$ is given by $\frac{1}{2}aq-M$. Our task, therefore, is to show that triangles DJE and EHC have the same area, which is equivalent to showing that $b(a-p)=aq$ or, since $p = q$, that $b(a-p)=ap$.

From $\frac{p}{a}=\frac{b}{a+b}$, we have the following:

$\frac{p}{a}=\frac{b}{a+b} \Rightarrow ab=p(a+b) \Rightarrow ab=pa+pb \Rightarrow pa=ab-pb \Rightarrow b(a-p)=ap$, which is the result we sought.