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FIRST-LEVEL EXAMINATION

Thursday, October 22, 2009

SOLUTIONS

1. The value of $\sqrt{\dfrac{5^2-4^2+3^2}{13^2-12^2+5^2}}$ **is** $\mathbf{\dfrac{3}{5}}$.

Solution: $\sqrt{\dfrac{5^2-4^2+3^2}{13^2-12^2+5^2}}=\sqrt{\dfrac{25-16+9}{169-144+25}}=\sqrt{\dfrac{2\bullet 9}{2\bullet 25}}=\dfrac{3}{5}$.

2. The operation $a\oplus b$ means a^2b-b^2a. Compute $(4\oplus 3)\oplus(3\oplus 2)$. **The answer is 432.**

Solution: $(4\oplus 3)=16\bullet 3-9\bullet 4=48-36=12$. $(3\oplus 2)=9\bullet 2-4\bullet 3=18-12=6$.

$(12\oplus 6)=144\bullet 6-36\bullet 12=864-432=432$.

3. Tomás drives 32,000 miles a year as a sales rep for an auto parts company. He wants to buy a hybrid car to get better gas mileage (48 mpg!), but the model he wants costs $4200 more than its non-hybrid counterpart (32 mpg). If gas costs $3.00 per gallon, about how long would it take Tomás to recoup the higher cost of the hybrid? **The answer is 50 months.**

Solution: The hybrid car will use $666\frac{2}{3}$ gallons per year, at a cost of $2000 in gas, whereas the non-hybrid car will use 1000 gallons per year, at a cost of $3000. The difference in the cost of gas ($1000 per year) will be made up in 4.2 years, which is about 50 months.

4. Given: $(2x+5)$ varies directly as $(y+3)^2$ and inversely as $(z-2)$, and x = 5 when y = 2 and z = 7. Find x when y = –1 and z = 6. **The answer is –1.**

Solution: $$(2x+5)=k\frac{(y+3)^2}{(z-2)}\Rightarrow 15=k\left(\frac{25}{5}\right)\Rightarrow k=3.$$

$$(2x+5)=3\frac{(y+3)^2}{(z-2)}\Rightarrow(2x+5)=3\left(\frac{4}{4}\right)\Rightarrow 2x=-2\Rightarrow x=-1.$$

5. The product of 99 and an integer k is 50x8x, where x represents a digit in the five-digit product. Find the integer k. **The answer is 513.**

Solution: If 50x8x is divisible by 99, it is divisible by 9, so its digits must sum to a multiple of 9. This means that x must equal 7. $\frac{50797}{99} = 513$.

6. If $63_{(x)} = 36_{(x+6)}$, then $x_{(10)} \cdot (x+2)_{(10)} = pq_{(12)}$. Find the base 10 sum $p + q$. **The answer is 8.**

Solution: $63_{(x)} = 36_{(x+6)} \Rightarrow 6x+3 = 3(x+6)+6 \Rightarrow 6x+3 = 3x+24 \Rightarrow x = 7$.

$(7 \bullet 9)_{10} = 63_{10} = 5(12)+3 = 53_{12} \Rightarrow p+q = 8$.

7. The five-digit number ABCDE has only even digits, such that A cannot be zero and the other digits may repeat. The four-digit number FGHJ has only odd digits, which may also repeat. How many ways can these numbers be configured so that ABCDE is twice FGHJ? **The answer is 0.**

Solution: The largest four-digit number with odd digits is 9999. Since twice 9999 is 19,998, any five-digit number ABCDE must have A = 1.

8. In right ΔABC, $\overline{AC} \perp \overline{BC}$, $AC = 192$, and $BC = 56$. $\overline{PD}$ is the perpendicular bisector of $\overline{AB}$, where $P \in \overline{AB}$, and $D \in \overline{AC}$. Compute PD. **The answer is $\mathbf{29\frac{1}{6}}$.**

Solution: ΔABC is a 7–24–25 triangle with hypotenuse 200. ΔADP is a similar triangle whose long leg has length PA = 100 (half the hypotenuse of ΔABC). The length of the short leg of ΔADP is therefore $PD = \frac{7}{24}(100) = \frac{700}{24} = 29\frac{1}{6}$.

9. Find the sum of all the integers in the following list (which omits multiples of 4):

1, 2, 3, 5, 6, 7, 9, …, 75, 77, 78, 79. **The answer is 2400.**

Solution: The sum of the first 79 positive integers is (after the method of Gauss) $39(80)+40=3160$. The sum of the first 19 multiples of 4 is $9(80)+40=760$. Subtracting this sum from the first sum leaves 2400.

10. If $P(x+3)=x^2+7x+4$, and $P(x)=ax^2+bx+c$, find the ordered triple (a, b, c). **The answer is (1, 1, –8).**

Solution: Let $y=x+3$. Then, $x=y-3$. Therefore, $P(x+3)=P(y-3+3)=(y-3)^2+7(y-3)+4=y^2+y-8$. Thus, $P(y)=y^2+y-8$, or, equivalently, $P(x)=x^2+x-8 \Rightarrow (a,b,c)=(1,1,-8)$

11. Given the following array, name the third number in the 20^{th} row. (Note: 11 is the second number in the third row.) **The answer is 727.**

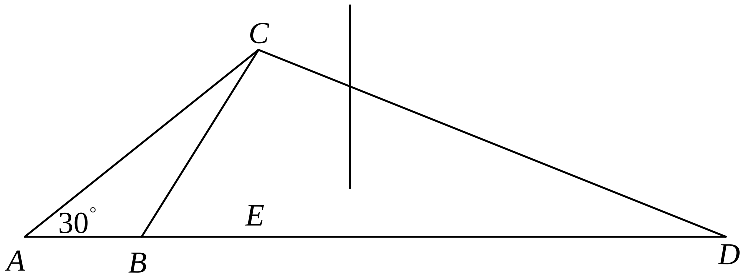
				1				
			3	5	7			
		9	11	13	15	17		
	19	21	23	25	27	29	31	
33	35	37	39	41	43	45	47	49

Solution: The number of numbers in each row follows the sequence 1, 3, 5, 7, …. In 19 rows, therefore, there are 1 + 3 + 5 + … + 37 = 361 numbers, of which the last is 721 (the 361^{st} odd number). The next three odd numbers are 723, 725, and 727.

12. In the figure below, $AB = 12\sqrt{3}$, and the area of ΔABC is $90\sqrt{3}$. Find $\cot \angle CBD$. **The answer is $\frac{\sqrt{3}}{5}$.**

Solution: If $AB = 12\sqrt{3}$ and the area of ΔABC is $90\sqrt{3}$, then the height of ΔABC is 15. Thus, in right Δ *ACE* below, *CE* = 15 and *AE* = $15\sqrt{3}$, so that $BE = 3\sqrt{3}$. Therefore,

$$\cot \angle CBD = \cot \angle CBE = \frac{3\sqrt{3}}{15} = \frac{\sqrt{3}}{5}.$$



13. An octahedral die has eight faces, numbered 1 through 8. Ayesha rolls a fair octahedral die three times. What is the probability that the number on the third roll is the product of the numbers from the first two rolls? **The answer is $\frac{5}{128}$.**

Solution: Three rolls of a die produce 512 outcomes: (1, 1, 1), (1, 1, 2), (1, 1, 3), etc. Of these outcomes, how many are such that the product of the first two rolls equals the third? The following table lists all the possible ordered triples that meet this criterion.

(1, 1, 1)	(1, 2, 2)	(2, 1, 2)	(1, 3, 3)	(3, 1, 3)
(1, 4, 4)	(2, 2, 4)	(4, 1, 4)	(1, 5, 5)	(5, 5, 1)
(1, 6, 6)	(2, 3, 6)	(3, 2, 6)	(6, 1, 6)	(1, 7, 7)
(7, 1, 1)	(1, 8, 8)	(2, 4, 8)	(4, 2, 8)	(8, 1, 8)

14. Let *f* be a function defined for all real numbers with the property that $f(3-x) = f(3+x)$. Suppose that f has 6 roots. Find the sum of the roots. **The answer is 18.**

Solution: If $3+x_0$ is a solution, then $3-x_0$ is also a solution. Therefore, the average value of the roots is 3. And $3 \bullet 6 = 18$.

15. Find the positive integral values of *x* and *y* such that $x+2y=20$, and $13x+11y$ is a multiple of 17. **The answer is (8, 6).**

Solution: The positive integral solutions to $x+2y=20$ are (2, 9), (4, 8), (6, 7), (8, 6), (10, 5), (12, 4), (14, 3), (16, 2), and (18, 1). The first of these ordered pairs, when substituted into $13x+11y$, produces 125, which is not a multiple of 17. Each successive ordered pair increases the sum $13x+11y$ by 15. A multiple of 17 is achieved with the ordered pair (8, 6).

16. Find the numerical value of the 21st term of the geometric sequence whose first three terms are $(1+i)^{20}$, $(1-i)^{16}$, $(2i)^6$. **The answer is $-\frac{1}{2^{30}}$.**

Solution: $(1+i)^{20} = \left[(1+i)^2\right]^{10} = \left[1+2i+i^2\right]10 = [2i]^{10} = 2^{10} \bullet i^{10} = -2^{10}$.

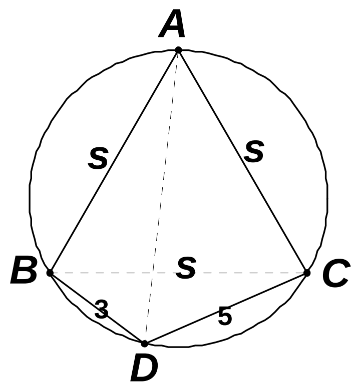
$$(1-i)^{16} = \left[(1-i)^2\right]^8 = \left[1-2i+i^2\right]^8 = [-2i]^8 = -2^8 \bullet i^8 = -2^8.$$

$$(2i)^6 = 2^6 \bullet i^6 = -2^6$$

Thus, the terms are $-2^{10}, -2^8, -2^6, \ldots$. The 21st term of this sequence is $-2^{-30} = \frac{1}{-2^{30}}$.

17. Equilateral triangle *ABC* is inscribed in a circle. *D* is a point on minor arc $\overset{\frown}{BC}$. The length of chord $\overline{BD}$ is 3 inches and the length of chord $\overline{DC}$ is 5 inches. How long is *AD* (in inches)? **The answer is 8.**

Solution: In the figure, the lengths of the sides of the triangle are given by *s*. According to Ptolemy's theorem[1], $3s+5s=(AD)s$, so that *AD* = 8.



Alternatively, we may recognize that, as inscribed angles, $\mathrm{m}\angle BDA=\mathrm{m}\angle CDA=60°$, and then use the Law of Cosines with ΔBCD and ΔACD.

In ΔBCD, $s^2=3^2+5^2-2\cdot3\cdot5\cos(120°)=9+25+15=49$, so that *s* = 7. Then, in ΔACD, we have $s^2=49=5^2+(AD)^2-2\cdot5(AD)\cos(60°)$, which, upon rearrangement, becomes the quadratic $(AD)^2-5(AD)-24$, which has roots –3 and 8. The former is, of course, extraneous.

[1] Ptolemy's theorem states that in a cyclic quadrilateral, the sum of the products of the lengths of the pairs of opposite sides equals the product of the lengths of the diagonals.

18. Two values of x make the following statement true. Find the absolute value of their difference. **The answer is $\frac{3}{8}$**

$$\left(\log_{\sqrt{3}} 5\right)\left(\log_{125} 9\right) + \log_2 \sqrt[3]{x} + \log_x 2 = 0$$

Solution. Using the change-of-base rules for logarithms:

$$\log_{\sqrt{3}} 5 = \frac{\log 5}{\log 3^{\frac{1}{2}}} = \frac{\log 5}{\frac{1}{2}\log 3} = 2\frac{\log 5}{\log 3}.$$

$\log_{125} 9 = \frac{\log 3^2}{\log 5^3} = \frac{2\log 3}{3\log 5}$. So $\left(\log_{\sqrt{3}} 5\right)\left(\log_{125} 9\right) = \frac{4}{3}$.

$\log_2 \sqrt[3]{x} = \frac{\log x^{\frac{1}{3}}}{\log 2} = \frac{\log x}{3\log 2}$, and $\log_x 2 = \frac{\log 2}{\log x}$. So the original equation becomes:

$$\frac{4}{3} + \frac{\log x}{3\log 2} + \frac{\log 2}{\log x} = 0 \Rightarrow \frac{4\log 2\log x + \left(\log x\right)^2 + 3\left(\log 2\right)^2}{3\log 2\log x} = 0.$$

The numerator of this last fraction is a quadratic that factors to $\left(\log x + 3\log 2\right)\left(\log x + \log 2\right)$. Thus:

$\log x = -3\log 2 \Rightarrow x = \frac{1}{8}$, and $\log x = -\log 2 \Rightarrow x = \frac{1}{2}$.

The positive difference of these two values of x is $\frac{3}{8}$.

19. The positive difference between the square of the arithmetic mean of two numbers and the square of their geometric mean is 36. Find the positive difference between the arithmetic mean of their squares and the square of their arithmetic mean. **The answer is 36.**

Solution. Let x and y be the two numbers, and let $x > y$. Then, the first of the above conditions produces

$\left(\frac{x+y}{2}\right)^2 - \left(\sqrt{xy}\right)^2 = \frac{x^2+2xy+y^2}{4} - xy = 36 \Rightarrow x^2+2xy+y^2-4xy=144$. This simplifies to $(x-y)^2 = 144 \Rightarrow x-y=12$, because of the requirement that $x > y$.

The second of the above conditions produces

$\frac{x^2+y^2}{2} - \frac{x^2+2xy+y^2}{4} \Rightarrow \frac{2x^2+2y^2-x^2-2xy-y^2}{4}$, which simplifies to $\frac{(x-y)^2}{4}$.

Because $x - y = 12$, this last expression equals 36.

20. In ΔABC, A is at (2, 4), B is at (8, 12), and C is at (16, 6). Find the coordinates (x, y) of point K on the segment $\overline{AB}$ so that the ratio of the area of ΔBKC to the area of ΔABC is 1:5. **The answer is** $\left(\mathbf{\frac{34}{5}}, \mathbf{\frac{52}{5}}\right)$**.**

Solution. ΔABC is an isosceles right triangle of leg length 10 and area 50. Therefore, we want K to be located on $\overline{AB}$ such that $KB = \frac{1}{5}AB$. The coordinates of K are given by

$\left(2+\frac{4}{5}(8-2),\ 4+\frac{4}{5}(12-4)\right) = \left(\frac{34}{5}, \frac{52}{5}\right)$.

21. The country of Halfway issues coins in denominations of 8, 9, and 10 halfmarks, the unit of currency. What is the largest number of halfmarks that cannot be expressed with these coins? **The answer is 31.**

Solution. We want to find a string of eight consecutive numbers of halfmarks that can be expressed with these denominations. Once we have that string, we can express the next eight by increasing by adding one halfmark coin to each of the first eight numbers. Each new string of eight can be expressed in a similar manner.

Obviously, we can express 8, 9, and 10 halfmarks with these coins. We can also express 16, 18, and 20 by using two of each coin, but also 17 (by using an 8 and a 9) and 19 (by using a 9 and a 10). Likewise, we can express 24–30 halfmarks—notice the pattern?—and then 32–40. This last is actually a string of nine, so 31 is the largest unexpressable number.

22. Let AB_9 and CD_9 be non-negative two-digit integers in base 9 (*A* and *C* may be zero), such that $(AB)^2(CD)-(AB)^2-(CD)=618_9$. For a particular solution (A_1, B_1, C_1, D_1), let $k_1 = A_1 + B_1 + C_1 + D_1$. Find the sum (in base 9) of all k_i. **The answer is 44.**

Solution: Adding 1 to both sides of the equation and factoring, we get $\left((AB)^2-1\right)(CD-1)=620_9$. In base 10, this is $6(9^2)+2(9)=504=2^3\cdot 3^2\cdot 7^1$, which has 4(3)(2) = 24 positive integer factors, or 12 factor pairs. Since one of the factors is $AB^2 - 1$, we are looking for a factor that is 1 less than a perfect square. The number 504 has five such factors: 3, 8, 24, 63, and 168.

Hence, in base 10, if $(AB)^2-1$ = 3, 8, 24, 63, or 168, *AB* = 2, 3, 5, 8, or 13. Of these, we discover that 2 is too small a value for our needs. The remaining numbers, in base 9, are 03, 05, 08, and 14. Correspondingly, if *CD* – 1 = 168, 63, 21, 8, or 3, then *CD* = 169, 64, 22, 9, or 4. Of these, we discover that 169 is *too large* a value for our needs. Again, the remaining numbers, in base 9, are 71, 24, 10, and 04.

Thus, (*A*, *B*, *C*, *D*) = (0, 3, 7, 1), (0, 5, 2, 4), (0, 8, 1, 0) and (1, 4, 0, 4), and the respective values of *k*, in base 9, are 12, 12, 10, and 10, which add to 44_9.

23. Find the polynomial of least positive degree that, when divided by $2x^2+3x-2$, $2x^2-3x+1$, or x^2+x-2, leaves a remainder of 5. Write your answer as a polynomial of least degree with integer coefficients. **The answer is $\mathbf{2x^3+x^2-5x+7}$.**

Solution. If $P(x)$ denotes the polynomial of least degree that we seek, we have

$$P(x)=\begin{cases}(2x^2+3x-2)(x-a)+5\\(2x^2-3x+1)(x-b)+5\end{cases}.$$ (Any pair of divisors would have worked here.)

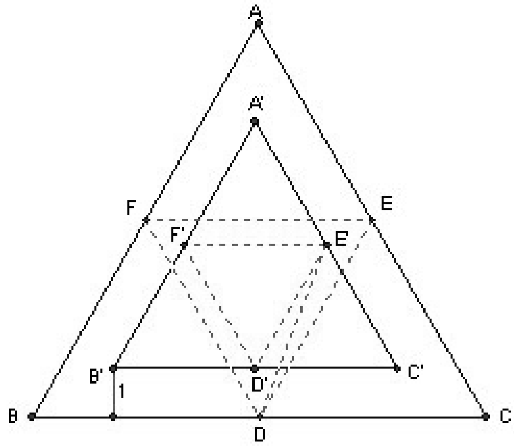
Equating the coefficients of x^2, we have $-2a+3=-2b-3 \Rightarrow a-b=3$.

Equating the coefficients of x, we have $-3a-2=3b+1 \Rightarrow a+b=-1$.

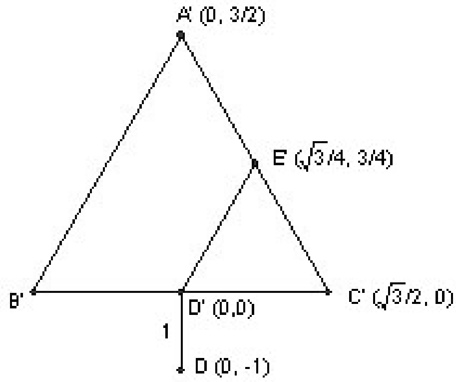
Solving this system, we have $(a, b) = (1, -2)$, and

$P(x)=(2x^2+3x-2)(x-1)+5=2x^3+x^2-5x+7$. We can verify that x^2+x-2 is a divisor of this polynomial.

24. Let ΔABC and $\Delta A'B'C'$ be equilateral triangles with the same center such that the distance between corresponding sides is 1 and each side of $\Delta A'B'C' = \sqrt{3}$. Let D, E, F, D', E', and F' be midpoints of the sides as pictured. A mosquito flies along a straight path from D to E to F to D to E' to F' to D' to E'. Find the length of the mosquito's flight. **The answer is** $\mathbf{6\sqrt{3} + \frac{\sqrt{13}}{2}}$**.**



Solution. In trapezoid $BB'C'C$, $m\angle B'BC = m\angle C'CB = 60°$, so the lengths of the sides of ΔABC are $3\sqrt{3}$. Thus, the paths represented by $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$ are midsegments whose combined length is $\frac{9\sqrt{3}}{2}$. Likewise, the paths represented by $\overline{E'F'}$, $\overline{F'D'}$, and $\overline{D'E'}$ are midsegments whose combined length is $\frac{3\sqrt{3}}{2}$. These six parts of the path thus have a combined length of $6\sqrt{3}$. This leaves $\overline{DE'}$. In the figure, D' is at the origin, and the other points are as shown. Applying the distance formula, we have $DE' = 13\sqrt{2}$.



25. Consider the set S of all positive fractions whose denominator is 24 and whose numerator is less than 26 and relatively prime with 24. How many nonempty subsets of this set have the property that the sum of all its elements is a reduced fraction? **The answer is 231.**

Solution: S is the 9-element set $\left\{\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24},\frac{25}{24}\right\}$.

S has 29 = 512 subsets, but those with an even number of elements will sum to $\frac{N}{24}$, where N will be an even number and this is a reducible fraction. Therefore, we restrict our attention to subsets with an odd number of elements. All 1-element subsets satisfy the requirement. We must look closely at 3, 5, 7 and 9-element subsets.

N must be a sum of an odd number of elements from {1, 5, 7, 11, 13, 17, 19, 23, 25} which is relatively prime with 24. Five of these numbers are of the form $3k + 1$, that is, they are 1 more than a multiple of 3. Call this set A = {1, 7, 13, 19, 25}.

Then, let B = {5, 11, 17, 23}, the set of numerators that are 2 more than a multiple of 3.

3-element subsets: Eliminate any that contain three elements from A (which add to a multiple of 3) or three elements from B (which add to a multiple of 6).

$$\binom{9}{3}-\binom{5}{3}-\binom{4}{3}=84-10-4=70$$

5-element subsets: Eliminate any that contain four elements from A and one element from B (which add to a multiple of 6) and any that contain one element from A and four elements from B (which add to a multiple of 9).

$$\binom{9}{5}-\binom{5}{4}\binom{4}{1}-\binom{5}{1}\binom{4}{4}=126-20-5=121$$

7-element subsets: Eliminate any that contain five elements from A and two elements from B (which add to a multiple of 9).

$$\binom{9}{7}-\binom{5}{5}\binom{4}{2}=36-6=30$$

9-element subsets: There is only one, namely S itself. This set produces a reduced fraction.

The total is 9 + 70 + 121 + 30 + 1 = **231**