# 46th ANNUAL

# MASSACHUSETTS MATHEMATICS OLYMPIAD

## 2009 – 2010

## A High School Competition Conducted by

## THE MASSACHUSETTS ASSOCIATION
## OF MATHEMATICS LEAGUES (MAML)

## And Sponsored by

## THE ACTUARIES' CLUB OF BOSTON

## FIRST-LEVEL EXAMINATION

## Thursday, October 22, 2009

1. The value of $\sqrt{\dfrac{5^2-4^2+3^2}{13^2-12^2+5^2}}$ is:

   a) 0
   b) $\dfrac{3}{5}$
   c) $\dfrac{\sqrt{3}}{5}$
   d) $\dfrac{5}{13}$
   e) $\dfrac{\sqrt{15}}{5}$

2. The operation $a\oplus b$ means $a^2b-b^2a$. Compute $(4\oplus3)\oplus(3\oplus2)$.

   a) 0
   b) 432
   c) 72
   d) 144
   e) 6

3. Tomás drives 32,000 miles a year as a sales rep for an auto parts company. He wants to buy a hybrid car to get better gas mileage (48 mpg!), but the model he wants costs $4200 more than its non-hybrid counterpart (32 mpg). If gas costs $3.00 per gallon, about how long would it take Tomás to recoup the higher cost of the hybrid?

   a) 36 months
   b) 40 months
   c) 45 months
   d) 50 months
   e) 60 months

4. Suppose that $(2x+5)$ varies directly as $(y+3)^2$ and inversely as $(z-2)$. If $y=2$ and $z=7$, then $x=5$. Find $x$ when $y=-1$ and $z=6$.

   a) 1
   b) −1
   c) 3
   d) −3
   e) −6

5. The product of 99 and an integer $k$ is $50x8x$, where $x$ represents a unique digit in the five-digit product. Find the integer $k$.

   a) 7
   b) 14
   c) 493
   d) 513
   e) 533

6.  If $63_{(x)} = 36_{(x+6)}$, then $x_{(10)} \cdot (x+2)_{(10)} = PQ_{(12)}$. Find the base 10 sum $P + Q$.

    a)  8            b)  9            c)  10            d)  11            e)  12

7.  The five-digit number ABCDE has only even digits, such that A cannot be zero and the other digits may repeat. The four-digit number FGHJ has only odd digits, which may also repeat. How many ways can these numbers be configured so that ABCDE is twice FGHJ?

    a)  0            b)  1            c)  2            d)  8            e)  24

8.  In right $\triangle ABC$, $\overline{AC} \perp \overline{BC}$, $AC = 192$, and $BC = 56$. $\overline{PD}$ is the perpendicular bisector of $\overline{AB}$, where $P \in \overline{AB}$, and $D \in \overline{AC}$. Compute $PD$.

    a)  $28\dfrac{5}{6}$        b)  $29\dfrac{1}{6}$        c)  28            d)  29            e)  24

9.  Find the sum of all the integers in the following list (which omits multiples of 4):

    1, 2, 3, 5, 6, 7, 9, …, 75, 77, 78, 79.

    a)  3160            b)  2320            c)  2400            d)  2440            e)  2520

10.  If $P(x+3) = x^2 + 7x + 4$, and $P(x) = ax^2 + bx + c$, find the ordered triple $(a, b, c)$.

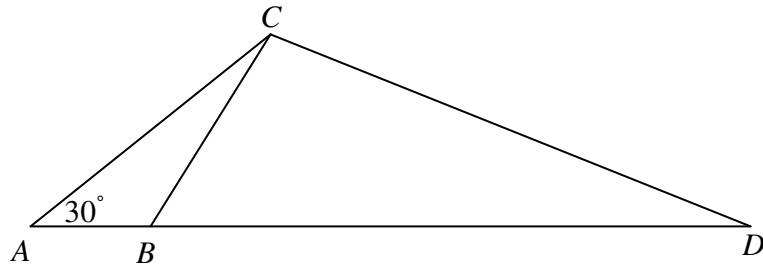    a)  (1, 7, 4)        b)  (1, –1, 8)        c)  (1, 1, –8)        d)  (4, 10, 7)        e)  (1, 7, –4)

11. Given the following array, name the third number in the $20^{th}$ row. (Note: 11 is the second number in the third row.)

$$1$$
$$3 \quad 5 \quad 7$$
$$9 \quad 11 \quad 13 \quad 15 \quad 17$$
$$19 \quad 21 \quad 23 \quad 25 \quad 27 \quad 29 \quad 31$$
$$33 \quad 35 \quad 37 \quad 39 \quad 41 \quad 43 \quad 45 \quad 47 \quad 49$$

a) 687  b) 697  c) 707  d) 717  e) 727

12. In the figure below, $AB = 12\sqrt{3}$, and the area of $\triangle ABC$ is $90\sqrt{3}$. Find $\cot \angle CBD$.



a) $\sqrt{3}$  b) $5\sqrt{3}$  c) $\dfrac{\sqrt{3}}{3}$  d) $\dfrac{4\sqrt{3}}{5}$  e) $\dfrac{\sqrt{3}}{5}$

13. An octahedral die has eight faces, numbered 1 through 8. Ayesha rolls a fair octahedral die three times. What is the probability that the number on the third roll is the product of the numbers from the first two rolls?

a) $\dfrac{5}{128}$  b) $\dfrac{7}{108}$  c) $\dfrac{9}{256}$  d) $\dfrac{1}{32}$  e) $\dfrac{21}{512}$

14. Let $f$ be a function defined for all real numbers with the property that

$f(3-x) = f(3+x)$. Suppose that $f$ has 6 roots. Find the sum of the roots.

a) 6       b) 12       c) 15       d) 18       e) 36

15. Let $x$ and $y$ be positive integers such that $x + 2y = 20$ and $13x + 11y$ is a multiple of 17.

Find the sum $x + y$.

a) 17       b) 16       c) 15       d) 14       e) 13

16. Find the numerical value of the $21^{\text{st}}$ term of the geometric sequence whose first three

terms are $(1+i)^{20}$, $(1-i)^{16}$, $(2i)^6$.

a) $-\dfrac{1}{2^{20}}$       b) $-2^{28}$       c) $-\dfrac{1}{2^{28}}$       d) $-\dfrac{1}{2^{30}}$       e) $-2^{30}$

17. Equilateral triangle $ABC$ is inscribed in a circle. $D$ is a point on minor arc $\overset{\frown}{BC}$. The

length of chord $\overline{BD}$ is 3 inches and the length of chord $\overline{DC}$ is 5 inches. How long is

$\overline{AD}$ (in inches)?

a) 3       b) $\sqrt{19}$       c) 6.5       d) 7       e) 8

18. Two values of $x$ make the following statement true. Find the absolute value of their

difference.

$$\left(\log_{\sqrt{3}} 5\right)\left(\log_{125} 9\right) + \log_2 \sqrt[3]{x} + \log_x 2 = 0$$

a) $\dfrac{3}{8}$       b) $\dfrac{3}{4}$       c) 2       d) 4       e) 6

19. The positive difference between the square of the arithmetic mean of two numbers and the square of their geometric mean is 36. Find the positive difference between the arithmetic mean of their squares and the square of their arithmetic mean.

    a) 12              b) 144              c) 72              d) 24              e) 36

20. In $\triangle ABC$, $A$ is at (2, 4), $B$ is at (8, 12), and $C$ is at (16, 6). Find the coordinates $(x, y)$ of point $K$ on the segment $\overline{AB}$ so that the ratio of the area of $\triangle BKC$ to the area of $\triangle ABC$ is 1:5.

    a) $\left(\dfrac{37}{5}, \dfrac{56}{5}\right)$     b) $\left(7, \dfrac{32}{3}\right)$     c) $(6, 10)$     d) $\left(\dfrac{16}{5}, \dfrac{28}{5}\right)$     e) $\left(\dfrac{34}{5}, \dfrac{52}{5}\right)$

21. The country of Halfway issues coins in denominations of 8, 9, and 10 halfmarks, the unit of currency. What is the largest number of halfmarks that cannot be expressed with these coins?

    a) 21              b) 23              c) 31              d) 55              e) 71

22. Let $AB_9$ and $CD_9$ be non-negative two-digit integers in base 9 ($A$ and $C$ may be zero), such that $(AB)^2(CD) - (AB)^2 - (CD) = 618_9$. For a particular solution $(A_1, B_1, C_1, D_1)$, let $k_1 = A_1 + B_1 + C_1 + D_1$. Find the sum (in base 9) of all $k_i$.

    a) 32              b) 36              c) 40              d) 44              e) 45

23. There is a polynomial $P(x)$ of least degree that, when divided by $2x^2 + 3x - 2$, $2x^2 - 3x + 1$, or $x^2 + x - 2$ leaves a remainder of 5. Which of the following is $P(x)$?

    a) $2x^3 + x^2 - 5x + 7$              b) $2x^3 + 7x^2 - 5x + 7$

    c) $2x^3 - x^2 + 5x + 7$              d) $2x^3 - 7x^2 - 5x - 7$

    e) $2x^3 + x^2 - 5x - 7$

24. Let $\triangle ABC$ and $\triangle A'B'C'$ be equilateral triangles with the same center such that the distance between corresponding sides is 1 and each side of $\triangle A'B'C' = \sqrt{3}$. Let $D, E, F, D', E',$ and $F'$ be midpoints of the sides as pictured. A mosquito flies along a straight path from $D$ to $E$ to $F$ to $D$ to $E'$ to $F'$ to $D'$ to $E'$. Find the length of the mosquito's flight.



a) $6\sqrt{3}$

b) $6\sqrt{3} + \sqrt{13}$

c) $12\sqrt{3} + \sqrt{26}$

d) $6\sqrt{3} + \dfrac{\sqrt{13}}{2}$

e) $12\sqrt{2} + \dfrac{\sqrt{13}}{2}$

25. Consider the set S of all positive fractions whose denominator is 24 and whose numerator is less than 26 and relatively prime to 24. How many nonempty subsets of S have the property that the sum of all the elements of the subset equals $N/24$, where $N$ is relatively prime to 24?

a) 231    b) 258    c) 8!    d) 128    e) 101

# 46th ANNUAL
# MASSACHUSETTS MATHEMATICS OLYMPIAD
# 2009–2010

1. *DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.*

2. This is a 25-question, multiple-choice test. Each question is followed by answers marked (a), (b), (c), (d), and (e). Only one of these is correct.

3. You are permitted to use scratch paper, graph paper, rulers, compasses, protractors, and erasers.

4. **CALCULATORS ARE NOT PERMITTED.** Students with a calculator near their desk, even if it is not used, will be disqualified.

5. Figures are not necessarily drawn to scale.

6. Mark your answer to each problem on the ParSCORE scoresheet with a #2 pencil. Check your blackened ovals for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded. The scoresheet has numbers up to 200; leave numbers 26 to 200 blank.

7. SCORING: You will receive 6 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer. While random guessing will almost certainly lower your score, guessing might be advantageous if you can eliminate more than two answers.

8. Before beginning the contest, your proctor will ask you to record specific information on the answer form. Specifically:

   - Your I.D. NUMBER is your school's six-digit CEEB code followed by a three-digit student ID number, of the form AAAAAABBB. Leave the last column blank.

   - For your NAME, enter your complete last name, skip one space, then enter as much of your first name as you can. If your entire first name fits, skip a space and enter your middle initial, if it fits. If your first name and middle initial do not fit, you will hand-write them elsewhere.

   - For your CODE, in the first column enter:
     A if you are in grade 8 or below
     B if you are in grade 9
     C if you are in grade 10
     D if you are in grade 11
     E if you are in grade 12
     F if you have graduated from high school (and are completing a postgraduate year)

   - Also for your CODE, in the second column enter Y if you were able to fit your complete name on the scoresheet; otherwise enter N. If you enter N, and score within the top 200 in the state, your handwritten name will be used, so **write legibly**. For example, the code "CY" specifies that you are a sophomore whose name fit completely; "EN" specifies that you are a senior whose name did not fit completely.
     – For TEST FORM, enter A.
     – Leave the PHONE NUMBER and SUBJECT CODE blank.
     – Turn your paper clockwise and carefully hand-write your name and today's date.
     – Underneath each letter or number, blacken the corresponding oval.

9. When your proctor gives the signal, begin working on the problems. You will have 90 MINUTES to complete the contest.