FORTY-FIFTH ANNUAL OLYMPIAD

HIGH SCHOOL PRIZE COMPETITION

IN MATHEMATICS

2008 – 2009

Conducted by

The Massachusetts Association
of
Mathematics Leagues
(MAML)

Sponsored by

The Actuaries' Club of Boston

SOLUTIONS

Tuesday, October 28, 2008

Solutions

1. Ans: **25**

Let x equal the number of juniors. Then $\dfrac{15 \cdot 95 + x \cdot 75}{15 + x} = 82.5$

→ $30(95) + 2(75x) = 165(x + 15)$ → $190 + 10x = 11(x + 15)$ → $x =$ **25**

2. Ans: **41**

Let (B, G) denote the original number of boys and girls, respectively. Then:

(1) $\dfrac{B}{G-5} = \dfrac{3}{2}$ and (2) $\dfrac{B-6}{G-5} = \dfrac{4}{3}$ From (1) we have $G - 5 = \dfrac{2B}{3}$

Substituting in (2) we have $\dfrac{B-6}{2B/3} = \dfrac{4}{3}$ → $3B - 18 = \dfrac{8B}{3}$ → $9B - 54 = 8B$ → $B = 54$

→ $G = \dfrac{\cancel{54}^{18}(2)}{\cancel{3}^{1}} + 5 =$ **41**

3. Ans: **– 180**

Let n be a typical number in set S.

According to the stated rule, n becomes $6(n - 6) + 6 = 6n - 30$ and the sum S becomes $6S - 30p$ → $(a, b) = (6, -30)$ → $ab = -180$

4. Ans: $-1 - \dfrac{1}{x}$

$$\frac{1 - \frac{2}{x} - \frac{3}{x^2}}{\frac{3}{x} - 1} = \frac{x^2 - 2x - 3}{x^2} \cdot \frac{x}{3 - x} = (-1) \cdot \frac{\cancel{(x-3)}(x+1)}{x^2} \cdot \frac{x}{\cancel{3-x}} = \frac{-x-1}{x} = -1 - \frac{1}{x}$$

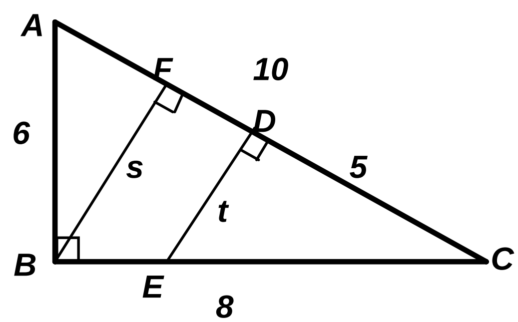
5. Ans: **19 : 22**

Since ΔABC is isosceles, $m\angle B = m\angle C = \dfrac{180 - 48}{2} = 66°$

→ ext angle at $A = 180 - 48 = 132°$ and ext angle at $C = 180 - 66 = 114°$

Thus, the required ratio is 114 : 132 = 57 : 66 = **19 : 22**

6. Ans: $\dfrac{21}{20}$



Since D is a midpoint, $DC = 5$.

$\Delta CDE \sim \Delta CBA$ → $\dfrac{t}{5} = \dfrac{6}{8}$ → $t = \dfrac{15}{4}$

The area of ΔCBA may be computed as

$\dfrac{1}{2} \cdot 6 \cdot 8$ or $\dfrac{1}{2} \cdot 10 \cdot s$, implying $s = \dfrac{6 \cdot 8}{10} = \dfrac{24}{5}$

Therefore, $FB - DE = \frac{24}{5} - \frac{15}{4} = \frac{96-75}{20} = \underline{\mathbf{\frac{21}{20}}}$

7. Ans: **777**

Let $10x + y$ denote the original two-digit number. Then:

$(10x + y)(xy) = \boxed{yyy} = 100y + 10y + y = 111y$

→ $(10x + y)(x) = 111 = 3(37)$

Thus, $x = 3$ and $10x + y = 37$ → $y = 7$ and the three digit number is **777**.

8. Ans: $\frac{7}{5}, -\frac{3}{2}$

1st pass: $4(2N) + 2N = 10N$

2nd pass: $10N(2N) + 2N = 42$ → $20N^2 + 2N - 42 = 0$

→ $10N^2 + N - 21 = (5N - 7)(2N + 3) = 0$ → $N = \underline{\mathbf{\frac{7}{5}, -\frac{3}{2}}}$

9. Ans: $\left(x^2+3x-2\right)(x-1)(x-2)$

$x^4 - 4 + 12x - 9x^2 = x^4 - (9x^2 - 12x + 4) = \left(x^2\right)^2 - (3x-2)^2 = \left(x^2+3x-2\right)\left(x^2-3x+2\right) =$

$\underline{\mathbf{\left(x^2+3x-2\right)(x-1)(x-2)}}$

10. Ans: 30

Let (G, D) denote the lengths of the giant's step and Jack's step (in feet), respectively.
Therefore, $2G = 11D$, and without any loss of generality, we may let $G = 11$ and $D = 2$.
In 1 unit of time, the giant takes 3 steps (33 feet) and the dwarf takes 8 steps (16 feet).
Suppose it takes T units of time for the giant to catch up to Jack who had an 85 - step head start. Then $85(2) + 16T = 33T$ → $T = 10$.
In 10 units of time, the giant takes **30** steps.

11. Ans: 18π

Since the ratio of the sides of the two cones is 1 : 2, then the ratio of the volumes of the cones is 1: 8. Since the volume of the larger cone is 144π cm^3, the volume of the smaller cone is $\frac{144}{8}\pi \text{ cm}^3 = 18\pi \text{ cm}^3$.

Alternate solution:

$\frac{1}{3}\pi R^2 h = 144\pi$ → $R^2 h = 432$

$h = 10$ → $R = \sqrt{\frac{216}{5}} = \frac{6}{5}\sqrt{30}$

By similar triangles, $R = 2r$ → $r = \frac{3}{5}\sqrt{30}$

Thus, $V = \frac{1}{3}\pi\left(\frac{3}{5}\sqrt{30}\right)^2 \cdot 5$ = **18π**

or since $V' = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi r^2 h'$ and $V = \frac{1}{3}\pi(2r)^2\left(2h'\right)$ → $\frac{V}{8} = \frac{1}{3}\pi r^2 h'$ ∴ $\frac{144\pi}{8}$ = **18π**

12. Ans: $y = \pm\frac{3}{4}x$

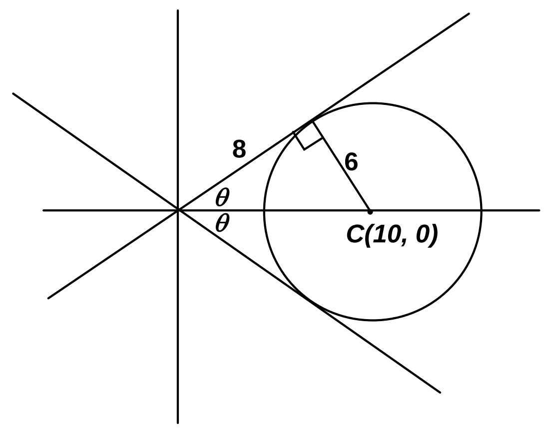
Completing the square we see that C_1 is a circle.

$\left(x^2 - 20x + \underline{100}\right) + y^2 = -64 + \underline{100}$

→ $(x-10)^2 + y^2 = 36$

This is a circle with center at (10, 0) and radius 6. Knowing that tangents to a circle are perpendicular to a radius drawn to the point of contact, we notice the 6 – 8 – 10 triangle and, therefore, tangents through the origin will have slopes of $\pm\frac{3}{4}$.

Thus, the equations are: $\boldsymbol{y = \pm\frac{3}{4}x}$



13. Ans: 6

$\log_6 6 + \log_6 x^2 + \log_6\left(36^2 x\right) = 8$ → $1 + \log_6 x^2 + 4 + \log_6 x = 8$

→ $\log_6 x^3 = 3$ → $x^3 = 6^3$ → x = **6**

14. Ans: 23

$$p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s}}} = 1 + \frac{21}{68}$$

If all variables are positive integers, then $p = 1$.

Eliminate p and invert both sides: $q + \cfrac{1}{r + \cfrac{1}{s}} = \frac{68}{21} = 3 + \frac{5}{21}$ → $q = 3$

Eliminate and invert again: $r + \frac{1}{s} = \frac{21}{5} = 4 + \frac{1}{5}$ → $r = 4$ and $s = 5$

Thus, $pq + rs = 1(3) + 4(5) =$ **23**

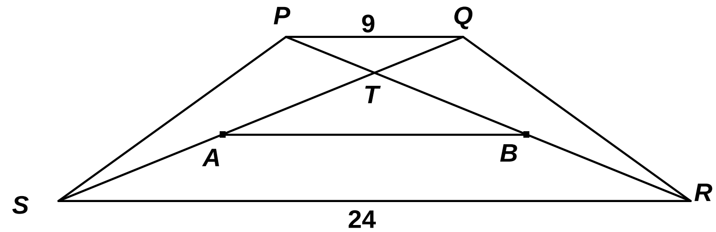
15. Ans: 13

$\Delta PQT \sim \Delta RST$ and the ratio of corresponding sides is $\frac{9}{24} = \frac{3}{8}$ → $PT = \frac{3}{11}PR$

But $PB = \frac{2}{3}PR$ → $BT = \left(\frac{2}{3} - \frac{3}{11}\right)PR = \frac{13}{33}PR$ → $\frac{PT}{BT} = \frac{3/11}{13/33} = \frac{9}{13}$

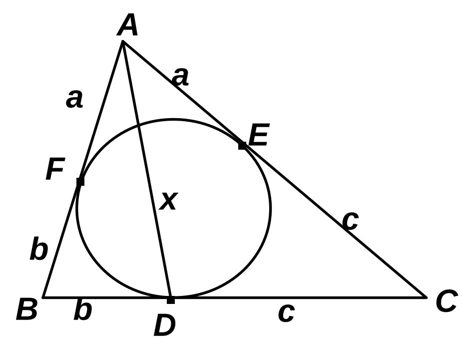
$SA : AQ = RB : BP = 1 : 3$ → $\Delta PQT \sim \Delta BAT$

$\therefore \frac{PQ}{BA} = \frac{PT}{BT}$ → $\frac{9}{BA} = \frac{9}{13}$ → $AB =$ **<u>13</u>**



16. Ans: 5

$$\begin{cases} a+b=5 \\ a+c=7 \\ b+c=6 \end{cases} \rightarrow (a, b, c) = (3, 2, 4)$$



Using Hero's formula to find the area of ΔABC,

$A = \sqrt{9(2)(3)(4)} = 6\sqrt{6}$

Using $\frac{1}{2}ab\sin\theta$,

$A = \frac{1}{2}\cdot 6\cdot 5\cdot \sin B = 6\sqrt{6}$ → $\sin B = \frac{2\sqrt{6}}{5}$ → $\cos B = \frac{1}{5}$ or

using the law of cosines directly on ΔABC, $49 = 25 + 36 - 2\cdot 5\cdot 6\cdot\cos B = 61 - 60\cos B$ → $\cos B = \frac{1}{5}$.

Now in ΔABD: $x^2 = 2^2 + 5^2 - 2\cdot 2\cdot 5\cdot\cos B = 4 + 25 - 4 = 25$ → $x =$ **<u>5</u>**

17. Ans: 188

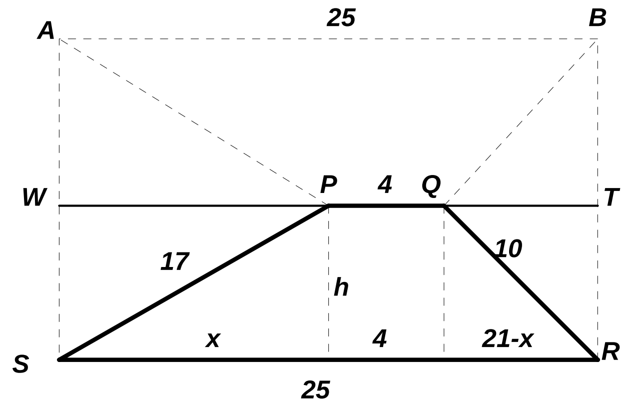
Let $P(n)$ denote the product $\left(1+\frac{2}{3}\right)\left(1+\frac{2}{4}\right)\left(1+\frac{2}{5}\right)\left(1+\frac{2}{6}\right)\left(1+\frac{2}{7}\right)\cdot\ldots\cdot\left(1+\frac{2}{46}\right)\cdot\ldots\cdot\left(1+\frac{2}{n}\right)$,

where $n \geq 3$. $P(3) = \frac{5}{3}$, $P(4) = \frac{5}{3}\cdot\frac{6}{4}$, $P(5) = \frac{\boxed{5}}{3}\cdot\frac{6}{4}\cdot\frac{7}{\boxed{5}}$, $P(6) = \frac{\boxed{6}}{3}\cdot\frac{7}{4}\cdot\frac{8}{\boxed{6}}$,

$P(7) = \frac{\boxed{7}}{3}\cdot\frac{8}{4}\cdot\frac{9}{\boxed{7}}$ …. In general, $P(n) = \frac{(n+1)(n+2)}{3\cdot 4}$

Therefore, $P(46) = \frac{47\cdot 48}{3\cdot 4} = 47(4) =$ **<u>188</u>**

18. Ans: 1152π



$x^2 + h^2 = 17^2$ and $h^2 + (21-x)^2 = 10^2$

Thus, $h^2 = 17^2 - x^2 = 10^2 - (21-x)^2$ and

$289 - x^2 = 100 - 441 + 42x - x^2$

→ $630 = 42x$ → $x = 15$ → $h = 8$

The rotation produces a cylinder with a circular base whose diameter is $\overline{BR}$ and whose height is $\overline{AB}$. Therefore, volume = $\pi(8)^2(25) = 1600\pi$.

However, we must eliminate two cones that have equal base diameters ($\overline{AS}$ and $\overline{BR}$), but different heights $\overline{PW}$ and $\overline{QT}$. From the diagram we see that $PW = 15$ and $QT = 6$ and the required volumes that must be deducted are:

$$\frac{1}{3}\pi(8)^2 6 + \frac{1}{3}\pi(8)^2 15 = \frac{1}{3}\pi(64)(21) = 448\pi$$

Therefore, the net volume is **<u>1152π</u>**.

19. Ans: 6

At $t = 1$, the runners are at (0, 4) and (3, 10) → $d^2 = 45$

At $t = 2$, the runners are at (0, 8) and (6, 10) → $d^2 = 40$

At $t = 2.5$, the runners are at (0, 10) and (7.5, 10) → $d = 7.5$ → $d^2 = 56.25$

Clearly a minimum has occurred, but where?

Assume the minimum occurs at time t, when the runners are at $(0, 4t)$ and $(3t, 10)$.

Let's examine d^2.

$d^2 = (3t)^2 + (10-4t)^2 = 25t^2 - 80t + 100$

Completing the square, $25\left(t - \frac{8}{5}\right)^2 + 36$ and a minimum value occurs at $t = 8/5$.

Substituting, $d = \sqrt{\left(\frac{24}{5}\right)^2 + \left(\frac{18}{5}\right)^2} = \sqrt{\frac{576}{25} + \frac{324}{25}} = \sqrt{\frac{900}{25}} = \sqrt{36} = \underline{\mathbf{6}}$

20. Ans: 10

A, B, C, D and E agree to meet every 2, 3, 4, 5 and 6 days respectively.

There are ${}_5C_3$ = 10 different threesomes.

{2, 3, 6} meet every 6 days (*ABE*)

{2, 3, 4}, {2, 4, 6}, {3, 4, 6} meet every 12 days (*ABC*, *ACE*, *BCE*)

{2, 4, 5} meet every 20 days (*ACD*)

{2, 3, 5}, {2, 5, 6}, {3, 5, 6} meet every 30 days (*ABD*, *ADE*, *BDE*)

{3, 4, 5}, {4, 5, 6}meet every 60 days (*BCD*, *CDE*)

Thus, we look at:

multiples of 6 that are not multiples of 12, 20, 30 or 60 - 6, 18, 42, 54, 66, 78

multiples of 20 that are not multiples of 6, 12, 30 or 60 – 20, 40, 80, 100
There are no multiples of 12 that are not multiples of 6, 20, 30 and 60 and likewise for multiples of 30 and multiples of 60.
Summing, $K = 6 + 4 = \underline{\mathbf{10}}$.

21. Ans: $(5,1), \left(\frac{1+i\sqrt{11}}{2}, \frac{-7+i\sqrt{11}}{2}\right)$ and $\left(\frac{1-i\sqrt{11}}{2}, \frac{-7-i\sqrt{11}}{2}\right)$

$$\begin{cases} x^3 - x^2y + 5xy^2 - y^3 = 124 \\ x^3 - 5x^2y + xy^2 - y^3 = 4 \end{cases}$$

$$(+) 2x^3 - 6x^2y + 6xy^2 - 2y^3 = 128$$
$$\leftrightarrow x^3 - 3x^2y + 3xy^2 - y^3 = 64$$
$$\leftrightarrow (x-y)^3 = 64 \leftrightarrow x - y = 4 \leftrightarrow x = y + 4$$
$$(-) 4x^2y + 4xy^2 = 120 \leftrightarrow xy(x+y) = 30$$

Substituting, $(y + 4)y(y + 4 + y) = 30 \rightarrow (y+4)y(y+2) = 15 \rightarrow y^3 + 6y^2 + 8y - 15 = 0$
By synthetic division we see that $y = 1$ is a root and we have a quadratic factor.

	1	6	8	−15
		1	7	15
$y=1$	1	7	15	0

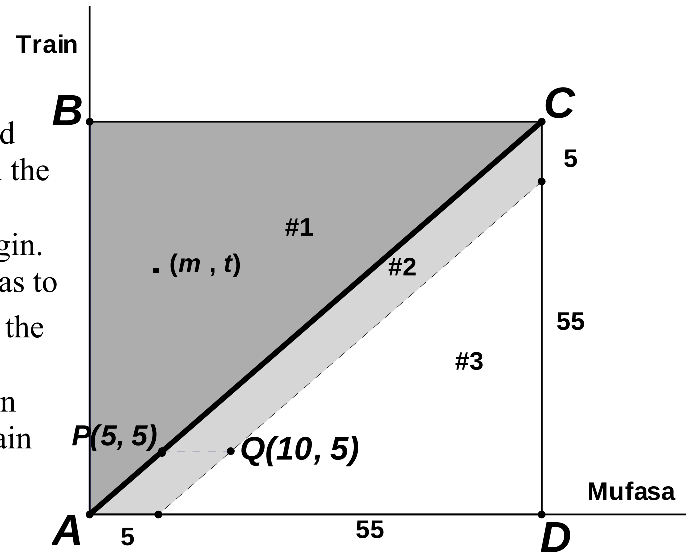
$y = 1 \rightarrow x = 5$

$$y^2 + 7y + 15 = 0 \rightarrow y = \frac{-7 \pm \sqrt{49-60}}{2} = \frac{-7 \pm i\sqrt{11}}{2} \rightarrow x = \frac{1 \pm i\sqrt{11}}{2}$$

Thus, the solution set contains the 3 ordered pairs:

$$\underline{\mathbf{(5,1), \left(\frac{1+i\sqrt{11}}{2}, \frac{-7+i\sqrt{11}}{2}\right) \text{and} \left(\frac{1-i\sqrt{11}}{2}, \frac{-7-i\sqrt{11}}{2}\right)}}$$

22. Ans: $\frac{167}{288}$

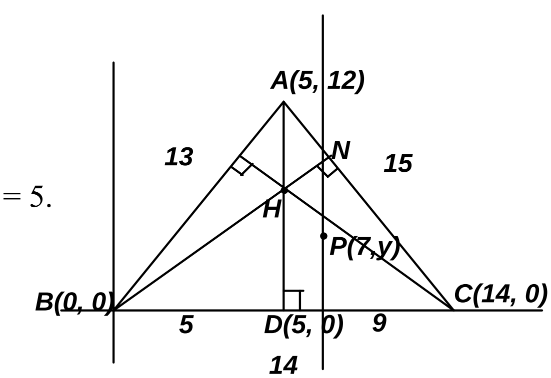


Let *ABCD* represent all of the times that Mufasa and the train can arrive at the station. Mufasa can catch the train at all points satisfying $m - t > 5$ that lie in the square. Let the lower right corner represent the origin. In region #1, Mufasa arrives before the train (and has to wait 😐). Along $\overline{AC}$, the train and Mufasa arrive at the same time. In region #2, Mufasa arrives within 5 minutes of the train (and gets to board ☺). In region #3, Mufasa arrives more than 5 minutes after the train (and misses his ride☹).

Hence, the probability is

$$1 - \frac{\frac{1}{2}55^2}{60^2} = \mathbf{\frac{167}{288}}$$

23. Ans: 273



The slope of $\overline{AC}$ is $-\frac{4}{3}$ → slope of $\overline{BN}$ is $\frac{3}{4}$

The equation of $\overline{BN}$ is $y = \frac{3}{4}x$ and the equation of $\overline{AD}$ is $x = 5$.

Therefore, the coordinates of *H* are $\left(5, \frac{15}{4}\right)$.

The area of ΔABC can be computed using Hero's formula.

Area = $\sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21(8)(7)(6)} = 84$

The radius *R* of the circumscribed circle can be computed as $\frac{abc}{4A} = \frac{13(14)(15)}{4(84)} = \frac{65}{8}$.

Since the center of the circumscribed circle lies on the point of concurrency of the perpendicular bisectors of the sides of the triangle, this center lies on the line $x = 7$, say at the point $P(7, y)$, where $y > 0$. Therefore, $R = \frac{65}{8} = PB = \sqrt{7^2 + y^2}$

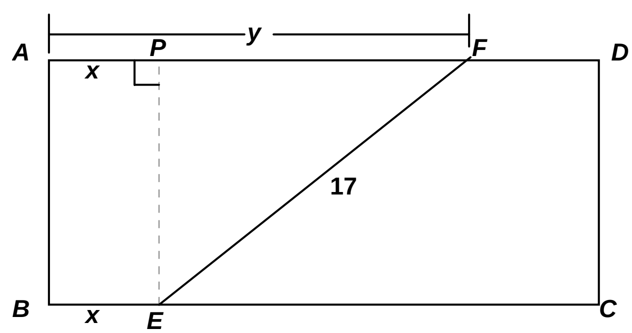
→ $y^2 = \frac{65^2 - 56^2}{8^2} = \frac{33^2}{8^2}$ → $y = \frac{33}{8}$

Alternate approach to finding $y = \frac{33}{8}$.

By finding the perpendicular bisectors of side $\overline{BC}$ $(x=7)$ and the perpendicular bisector of $\overline{AC}$. Midpoint of $\overline{AC}$ is $\left(\frac{19}{2},6\right)$, slope of $\overline{AC}=\frac{12-0}{-9}=-\frac{4}{3}$. Therefore the slope perpendicular bisector of AC is $\frac{3}{4}$. The equation of the perpendicular bisector is $y-6=\frac{3}{4}\left(x-\frac{19}{2}\right)$ or $y=\frac{3}{4}\left(\frac{2x-19}{2}\right)+6$. The point of intersection of $x=7$ and $y=\frac{3}{4}\left(\frac{2x-19}{2}\right)+6$ is $y=\frac{3}{4}\left(\frac{2\cdot 7-19}{2}\right)+6=\frac{3}{4}\left(-\frac{5}{2}\right)+6=-\frac{15}{8}+6=\frac{33}{8}$.

Thus, the required distance is $\sqrt{(7-5)^2+\left(\frac{33}{8}-\frac{15}{4}\right)^2}=\sqrt{4+\left(\frac{3}{8}\right)^2}=\frac{\sqrt{265}}{8}$ → 265 + 8 = **273**

24. Ans: $4+2x$



Let $AF=y$. Then:

$$\frac{\frac{1}{2}w(x+y)}{\frac{1}{2}w(l-y+l-x)}=\frac{l-x}{x} \rightarrow \frac{(x+y)}{(2l-x-y)}=\frac{l-x}{x}$$

$\rightarrow x^2+xy=2l^2-lx-ly-2xl+x^2+xy$

$\rightarrow 0=2l^2-ly-3xl$

$l\neq 0 \rightarrow 0=2l-y-3x \rightarrow l=\frac{y+3x}{2}$

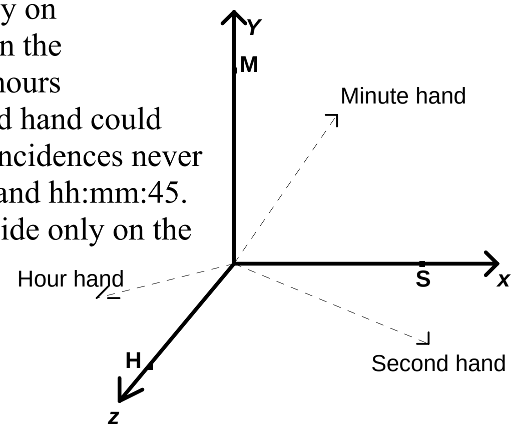
From the fact that both l and w are integers, we conclude that ΔEPF has sides of lengths 8, 15, and 17. Thus, $PF=8$ or 15 and $y=x+8$ or $x+15$.

Substituting, $l=\frac{8+4x}{2}=4+2x$ or $l=\frac{15+4x}{2}$

For integer values of x, the latter expression is never an integer and is rejected.

25. Ans: 26

In theory, the hour and the minute hands could coincide only on the positive or negative *y*-axis. Actual coincidences occur on the positive *y*-axis only. There are two such occurrences in 24 hours (at 3AM and 3PM). In theory, the hour hand and the second hand could coincide only on the positive or negative *z*-axis. Actual coincidences never occur since the second hand passes the *z*-axis at hh:mm:15 and hh:mm:45. In theory, the minute hand and the second hand could coincide only on the positive or negative *x*-axis. Actual coincidences occur only on the positive *x*-axis, which corresponds to 15 minutes past the hour. There are 24 such occurrences in 24 hours. Thus, the answer is 2 + 24 = **26**.



ANSWERS
1. D
2. E
3. A
4. D
5. C
6. A
7. D
8. A
9. E
10. E
11. A
12. D
13. B
14. D
15. D
16. A
17. A
18. B
19. B
20. B
21. E
22. E
23. B
24. D
25. D