

**FORTY-FIFTH ANNUAL OLYMPIAD
HIGH SCHOOL PRIZE COMPETITION
IN MATHEMATICS**

2008 – 2009

Conducted by

**The Massachusetts Association
of
Mathematics Leagues
(MAML)**

Sponsored by

The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

Tuesday, October 28, 2008

1. There are juniors and seniors in a math class. On the last test, the class average was 82.5%. The 15 seniors in the class had an average score of 95%. If the average for the juniors was 75%, how many juniors are there in the class?

(A) 9 (B) 15 (C) 20 (D) 25 (E) 40

2. After five girls leave a group of boys and girls, the ratio of boys to girls is 3:2. Shortly thereafter, six boys leave. Now the ratio of boys to girls is 4:3. How many girls were there originally?

(A) 17 (B) 23 (C) 29 (D) 33 (E) 41

3. A set of p numbers has a sum of S . Each number is decreased by 6, then multiplied by 6, then increased by 6. The sum of the new set of numbers is $aS + bp$. Find the product ab .

(A) -180 (B) 180 (C) 90 (D) 72 (E) -54

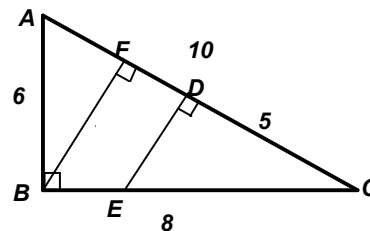
4. Simplify $\frac{1 - \frac{2}{x} - \frac{3}{x^2}}{\frac{3}{x} - 1}$

(A) $x + 1$ (B) $-x - 1$ (C) $\frac{x+1}{x}$ (D) $-1 - \frac{1}{x}$ (E) $\frac{x-1}{x}$

5. In $\triangle ABC$, $AB = AC$ and $m\angle A = 48^\circ$. Find, in lowest terms, the ratio of the measure of the exterior angle at vertex C to the exterior angle at vertex A .

(A) 8 : 11 (B) 11 : 8 (C) 19 : 22 (D) 22 : 19 (E) 1 : 1

6. In right $\triangle ABC$, $AB = 6$, $BC = 8$, $B = 90^\circ$, D is the midpoint of \overline{AC} , E lies on \overline{BC} and F lies on \overline{AC} such that $\overline{ED} \perp \overline{AC}$ and $\overline{BF} \perp \overline{AC}$. Find $BF - DE$.



(A) 21/20 (B) 5/4 (C) 7/5 (D) 31/20 (E) 2

7. A two-digit number is multiplied by the product of its digits, forming a three-digit YYY where Y is the units digit of the original number. Find the three digit number YYY .

(A) 111 (B) 333 (C) 555 (D) 777 (E) 999

8. A computing machine is designed to multiply whatever real number is fed into it by $2N$ and then to add $2N$ to the product. This number is then reported. The number 4 is fed into the machine and the number reported by the machine is x . The number x is then fed back into the machine. The machine now reports the number 42. What are the possible values of N ?

(A) $\frac{7}{5}, -\frac{3}{2}$ (B) $-\frac{7}{5}, -\frac{3}{2}$ (C) $\frac{7}{5}, \frac{3}{2}$ (D) $\frac{7}{5}, -\frac{2}{3}$ (E) $\frac{5}{7}, -\frac{3}{2}$

9. Factor $x^4 - 4 + 12x - 9x^2$ completely as a product of polynomials with integral coefficients.

(A) $(x^2 + 3x + 2)(x - 1)(x + 2)$ (B) $(x^2 - 3x - 2)(x - 1)(x - 2)$
(C) $(x^2 + 3x - 2)(x + 1)(x + 2)$ (D) $(x^2 - 3x + 2)(x - 1)(x - 2)$
(E) $(x^2 + 3x - 2)(x - 1)(x - 2)$

10. Jack is being pursued by a giant. For every 8 steps that Jack takes, the giant takes 3. In addition, 2 steps by the giant are as long as 11 steps by Jack. If Jack has a head start of 85 of his own steps, how many steps does the giant need to overtake Jack?

(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

11. A right circular cone has a height of 10 cm and a volume of $144\pi \text{ cm}^3$. If a plane intersects the cone perpendicular to the altitude and 5 cm from the vertex then what is the volume of the smaller cone that is determined by the cut?

(A) 18π (B) 30π (C) 36π (D) 54π (E) 72π

12. Find all equations of the lines through the origin which are tangent to the curve $C_1: x^2 - 20x + y^2 + 64 = 0$.

(A) $y = \pm \frac{4}{5}x$ (B) $y = \pm \frac{5}{4}x$ (C) $y = \pm 3x$
(D) $y = \pm \frac{3}{4}x$ (E) $y = \pm \frac{4}{3}x$

13. Find all values of x which satisfy the following equation:

$$\log_6 6x^2 + \log_{\sqrt{6}} 36\sqrt{x} = 8$$

(A) 3 (B) 6 (C) 9 (D) 18 (E) 36

14. Let p, q, r and s be positive integers and $p + \frac{1}{q + \frac{1}{r + \frac{1}{s}}} = \frac{89}{68}$.

Find the value of $pq + rs$.

- (A) 12 (B) 17 (C) 19 (D) 23 (E) 24

15. In trapezoid $PQRS$, \overline{PQ} is parallel to \overline{RS} , $PQ = 9$ cm and $RS = 24$ cm. Point A lies on diagonal \overline{QS} one-third of the way from S to Q , and point B lies on diagonal \overline{PR} two-thirds of the way from P to R . Find the length of \overline{AB} in centimeters.

- (A) 9 (B) 10 (C) 11 (D) 13 (E) 14

16. The circle inscribed in triangle ABC is tangent to \overline{BC} at D . The lengths of sides \overline{AB} , \overline{BC} and \overline{CA} are 5", 6" and 7", respectively. The length of \overline{AD} is x inches. Find x .

- (A) 5 (B) 6 (C) 5.5 (D) $6\sqrt{6}$ (E) $\frac{1}{5}$

17. Compute the value of $\left(1 + \frac{2}{3}\right)\left(1 + \frac{2}{4}\right)\left(1 + \frac{2}{5}\right)\left(1 + \frac{2}{6}\right)\left(1 + \frac{2}{7}\right) \cdot \dots \cdot \left(1 + \frac{2}{46}\right)$.

- (A) 188 (B) 190 (C) 94 (D) 95 (E) 55

18. In trapezoid $PQRS$, $\overline{PQ} \parallel \overline{RS}$, $PQ = 4$, $QR = 10$, $RS = 25$ and $PS = 17$. If the trapezoid is rotated about its shorter base, \overline{PQ} , what is the volume of the solid that is generated?

- (A) 1600π (B) 1152π (C) 448π (D) 2π (E) 216π

19. On a coordinate plane, Kevin is standing at $P(0, 10)$. Joba is standing at the origin. Kevin starts running due east at 3 feet per second while at the same time Joba starts running due north at 4 feet per second. What is the minimum distance between Kevin and Joba?

- (A) 5 (B) 6 (C) $\frac{25}{4}$ (D) 8 (E) 10

20. Five men agreed to meet on April 1st for lunch. Thereafter, each of them wanted to continue to have lunch together. Al said he would show up every second day. Bob said he would appear every third day. Charles said he would come every fourth day, Dan said he would come every fifth day, and Elton agreed to come every sixth day. During the next one hundred days, there were K days when only three men showed up. Find the value of K .

- (A) 6 (B) 10 (C) 4 (D) 12 (E) 54

21. Which of the ordered pairs (x, y) satisfy the system $\begin{cases} x^3 - x^2y + 5xy^2 - y^3 = 124 \\ x^3 - 5x^2y + xy^2 - y^3 = 4 \end{cases}$

I (5, 1)

II (5, -1)

III $\left(\frac{-1+i\sqrt{11}}{2}, \frac{7+i\sqrt{11}}{2}\right)$ and $\left(\frac{-1-i\sqrt{11}}{2}, \frac{7-i\sqrt{11}}{2}\right)$

IV $\left(\frac{1+i\sqrt{11}}{2}, \frac{-7+i\sqrt{11}}{2}\right)$ and $\left(\frac{1-i\sqrt{11}}{2}, \frac{-7-i\sqrt{11}}{2}\right)$

- (A) I only (B) II only (C) II, III (D) I, III (E) I, IV

22. A train stops at the train station at some random time between 1:00pm and 2:00pm. After five minutes, the train leaves. If Mufasa also arrives at the station at a random time between 1:00pm and 2:00pm, what is the probability that he will catch the train?

- (A) $\frac{121}{288}$ (B) $\frac{25}{60}$ (C) $\frac{1}{2}$ (D) $\frac{35}{60}$ (E) $\frac{167}{288}$

23. In $\triangle ABC$ the lengths of \overline{AB} , \overline{BC} , and \overline{CA} are 13, 14, and 15 inches, respectively. The altitudes of the triangle are concurrent at point H and \overline{AD} is the altitude to \overline{BC} . The distance from point H to the center of the circle that is circumscribed around $\triangle ABC$ may

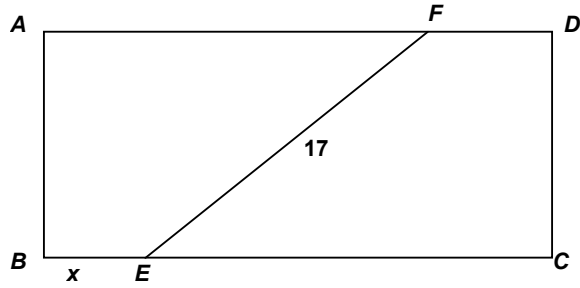
be written in simplified radical form as $\frac{\sqrt{p}}{q}$

Determine the sum $p + q$.

- (A) 265 (B) 273 (C) 285 (D) 108 (E) 72

24. Rectangle $ABCD$ has dimensions $AD = l$ and $AB = w$. $BE = x$ and $EF = 17$, $AF > BE$, and l , w , and x are all integers.

Find l in terms of x if $\frac{\text{area}(ABEF)}{\text{area}(ECDF)} = \frac{EC}{BE}$



- (A) $2 + 4x$ (B) $4 + 4x$ (C) $4 + 3x$ (D) $4 + 2x$ (E) $4 + x$
- 25 Victor created a clock in three dimensional space in the following way. At $t = 0$, the hour hand starts on the positive z -axis and rotates clockwise in the yz -plane when viewed from the positive x -axis. At $t = 0$ the minute hand starts on the positive y -axis and rotates clockwise in the xy -plane when viewed from the positive z -axis. At $t = 0$ the second hand starts on the positive x -axis and rotates clockwise in the xz -plane when viewed from the positive y -axis. Victor becomes melancholy when exactly two of his clock's hands coincide. How many times does he become melancholy in a 24 hour period?
- (A) 2 (B) 12 (C) 13 (D) 26 (E) 50

**Forty-Fifth Annual
MASSACHUSETTS MATHEMATICS OLYMPIAD
2008–2009**

1. ***DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.***
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the ParSCORE Score Sheet with a #2 pencil. Check your blackened ovals for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded. The scoresheet has numbers up to 200; leave numbers 26 to 200 blank.
4. SCORING: You will receive 6 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer. Note that random guessing will almost certainly lower your score. If you can eliminate more than two answers, guessing might be advantageous.
5. You are permitted to use scratch paper, graph paper, rulers, compasses, protractors, erasers and most calculators. No problems require the use of a calculator.
6. Calculators with a QWERTY keyboard, such as the TI-92 or HP-95, or which contain a Computer Algebra System, such as the TI-89, HP 40G, 48GII, HP49G, or HP50G, are not allowed and must be given to your proctor for the entirety of the contest. Students with such a calculator near their desk, even if it is not used, will be disqualified.
7. Figures are not necessarily drawn to scale.
8. Before beginning the contest, your proctor will ask you to record specific information on the answer form. Specifically:
 - Your I.D. NUMBER is your school's six-digit CEEB code followed by a three digit student ID number, of the form AAAAAABBB. Leave the last column blank.
 - For your NAME, enter your complete last name, skip one space, then enter as much of your first name as you can. If your entire first name fits, skip a space and enter your middle initial, if it fits. If your first name and middle initial do not fit, you will hand-write them elsewhere.
 - For your CODE, in the first column enter
 - A if you are in grade 8 or below
 - B if you are in grade 9
 - C if you are in grade 10
 - D if you are in grade 11
 - E if you are in grade 12
 - F if you have graduated from high school (and are completing a postgraduate year)
 - Also for your CODE, in the second column enter Y if you were able to fit your complete name on the scoresheet; otherwise enter N. If you enter N, and score within the top 200 in the state, your handwritten name will be used. For example, the code "CY" specifies that you are a sophomore whose name fit completely; "EN" specifies that you are a senior whose name did not fit completely.
 - For TEST FORM, enter A.
 - Leave the PHONE NUMBER and SUBJECT CODE blank.
 - Turn your paper clockwise and carefully hand-write your name and today's date. Underneath each letter or number, blacken the corresponding oval.
9. When your proctor gives the signal, begin working on the problems. You will have 90 MINUTES to complete the contest.