Forty-Fourth Annual

MASSACHUSETTS MATHEMATICS OLYMPIAD

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Conducted by

THE MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

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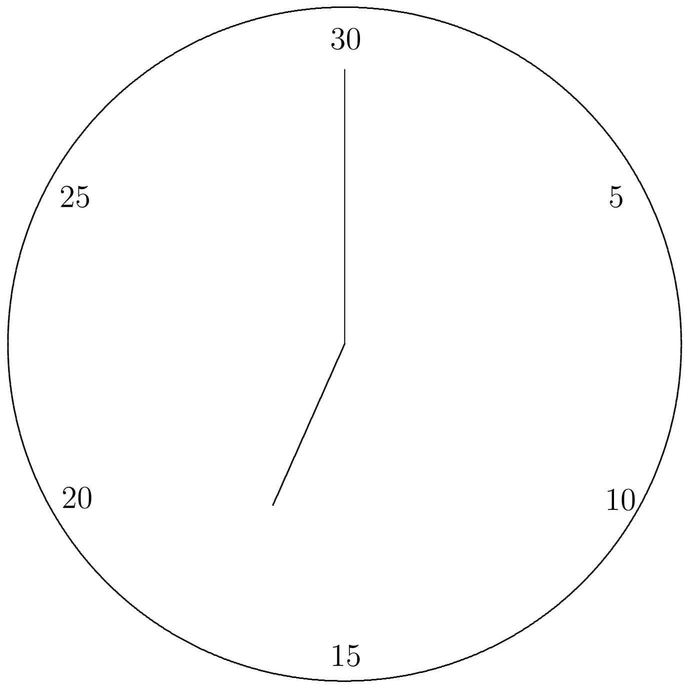
SECOND LEVEL EXAMINATION

TUESDAY, MARCH 4, 2008

1. **Time to begin.** As you know, a day is divided into 24 hours, each with 60 minutes. A confused watchmaker believes that a day has 60 hours, each with 24 minutes, and builds a fully-functional and accurate watch as follows:



The watch shows 17 o'clock. As with a standard watch, the numbers on the outside represent the hour; the hour hand makes two revolutions per day; and at "noon" and "midnight" (both 30:00 on the confused watch), both the hour and the minute hands point directly upward.

(a) (4 points) What will be the acute angle, in degrees, formed by the hands of the confused watch in 17 minutes — when it reads 17:17?

(b) (3 points) What time on the confused watch corresponds to 8:00 PM on a regular watch?

(c) (4 points) Between noon and midnight, how many times will the hands on the confused watch be perpendicular?

(d) (4 points) Suppose this watch were placed beside a regular watch (also fully functional and accurate). Between noon and midnight, how many times would the hands of the two watches point in the same directions? Exclude both noon and midnight.

2. **Studying hard.** A study hall is held in a classroom with a single row of n desks. To keep the students quiet, the strict teacher creates two rules: (1) no two students can sit in adjacent desks, and (2) once sitting no student can move.

   Students begin sitting, and after a certain amount of time, the study hall is "full" — in other words, no more students can sit without violating rules 1 or 2.

   For the following questions, when giving any diagrams, use "X" for a filled seat and "O" for an empty seat.

   (a) (2 points) Suppose $n = 15$. What is the smallest number of students who could be in the study hall when it is "full"? What is the largest number of students who could be in the study hall when it is "full"? Show an example of both cases.

   (b) (3 points) For each of $n = 1$ to $n = 7$, how many different ways can the study hall be "filled"? Each seat in the row is distinct (so two seating arrangements that are mirror images should both be counted).

   (c) (1 points) The number of seating arrangements for a certain number of desks n is a function of the number of seating arrangements for two smaller values of n. Find this recursive relationship. (There are two correct formulas possible. You need only state one.)

   (d) (2 points) Find the number of seating arrangements when $n = 15$.

   (e) (7 points) Prove the recursive relationship you identified in (c).

3. **So primitive, even a caveman could do it.** A Pythagorean triple is a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$. A triple is called *primitive* if and only if the greatest common divisor of its three integers is 1.

   (a) (2 points) Prove that for any positive integers p, q with $p > q$, that $(p^2 - q^2, 2pq, p^2 + q^2)$ is a Pythagorean triple (not necessarily primitive[1]).

   (b) (5 points) Explain why, in a primitive Pythagorean triple, the largest number is always odd, and one of the other numbers is always even.

   (c) (3 points) Define the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ by

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathcal{C} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

   If we rewrite familiar Pythagorean triples (a, b, c) in matrix form as $[a \quad b \quad c]$, show that the following matrix products are Pythagorean triples:

   i. $[3 \quad 4 \quad 5]\,\mathcal{A}$
   ii. $[3 \quad 4 \quad 5]\,\mathcal{B}$
   iii. $[3 \quad 4 \quad 5]\,\mathcal{C}$, and
   iv. $[5 \quad 12 \quad 13]\,\mathcal{B}$.

   (d) (5 points) If $[a \quad b \quad c]$ is a primitive Pythagorean triple, prove that the matrix product $[a \quad b \quad c]\mathcal{A}$ is also a primitive Pythagorean triple.

[1] A primitive triple will be formed iff p and q have a GCD of one and one of them is even.

4. **Algebra.**

(a) (5 points) Find the sum:

$$\sum_{b=2}^{100} \frac{1}{\log_b 100!}.$$

(b) (10 points) Show that there are infinitely many pairs of positive integers x, y, such that

$$x^{x-y} = y^{x+y}.$$

5. **A trigonometric identity.**

(a) (1 point) Prove the identity

$$\cos x = \frac{\sin 2x}{2\sin x}.$$

(b) (2 points) Prove the identity

$$\cos x + \cos 3x = \frac{\sin 4x}{2\sin x}.$$

(c) (3 points) Prove the identity

$$\cos x + \cos 3x + \cos 5x = \frac{\sin 6x}{2\sin x}.$$

(d) (4 points) Prove the identity

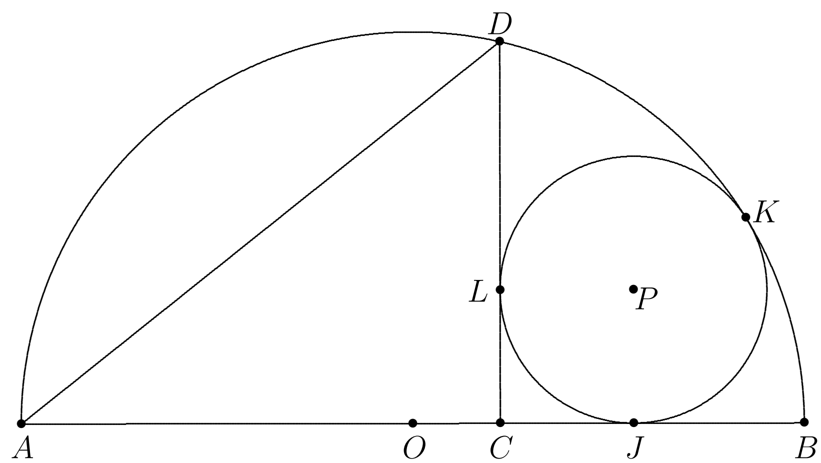
$$\cos x + \cos 3x + \cos 5x + \cos 7x = \frac{\sin 8x}{2\sin x}.$$

(e) (5 points / 15 points) Prove the identity

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin(2nx)}{2\sin x}.$$

(If you are able to prove this identity, you will receive full credit for the problem.)

6. **Geometry** (10 points) In the sketch below, $\overline{CD}$ is perpendicular to the diameter $\overline{AB}$ of the semicircle with center O. The inscribed circle with center P is tangent to $\overline{AB}$ at J, $\overline{CD}$ at L, and the semicircle at K. Show that the line segments $\overline{AD}$ and $\overline{AJ}$ have the same length.



7. **The absent-minded mathematician.**

(a) (1 point) A busy math teacher writes college recommendations for six students, each applying early-admission to a different college. In his haste to get to class, he stuffs each letter into one of the preaddressed envelopes and seals them. What is the probability that each letter is in the correct envelope?

(b) (4 points) What is the probability that *none* of the letters is in the correct envelope?

(c) (10 points) The same teacher has five pairs of socks, each a different shade of gray. He is too busy to sort his socks, so every Sunday after doing his laundry he randomly pairs up the ten socks, creating five pairs. Then every day from Monday to Friday he wears a different pair, so all five pairs are used during the week.

When the teacher wears socks that are either the same shade or differ by one shade, his students don't notice anything wrong. But when his socks differ by more than one shade, his students laugh at him. What is the probability that he makes it through the week without getting laughed at?