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MASSACHUSETTS MATHEMATICS OLYMPIAD
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FIRST LEVEL SOLUTIONS

TUESDAY, OCTOBER 23, 2007

MASSACHUSETTS MATHEMATICS OLYMPIAD
FIRST LEVEL ANSWER KEY
Tuesday, October 23, 2007

Question	**Answer**
1	D
2	A
3	A
4	D
5	C
6	B
7	D
8	C
9	E
10	C
11	B
12	A
13	A
14	A
15	B
16	E
17	C
18	C
19	B
20	E
21	E
22	D
23	A
24	D
25	C

MASSACHUSETTS MATHEMATICS OLYMPIAD FIRST LEVEL SOLUTIONS Tuesday, October 23, 2007

1. Allison draws five squares and eight triangles. How many sides do these shapes have altogether?

   (A) 13 **(B)** 26 **(C)** 39 **(D)** 44 **(E)** 47

   Answer: $\boxed{\text{D}}$

   Solution: The squares each have four sides, for a total of 20; and the triangles each have three, for a total of 24. Therefore there are $\boxed{44}$ sides altogether.

2. On my walk to school, I traveled at three different speeds: for the first mile, I traveled at 6 mph; for the second mile, I traveled at 2 mph; and for the final mile, I traveled at 3 mph. What was my average speed for the entire trip?

   (A) 3 mph **(B)** 3.5 mph **(C)** $3.\overline{6}$ mph **(D)** 4.25 mph **(E)** 5.5 mph

   Answer: $\boxed{\text{A}}$

   Solution: The total distance traveled was 3 miles, and the total time traveled was $\frac{1}{6}+\frac{1}{2}+\frac{1}{3}=1$ hour, for an average speed was $\boxed{\text{3 mph.}}$ More generally, the answer is the harmonic mean of 2, 3, and 6, i.e., the reciprocal of the average of the reciprocals:

$$\left(\frac{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}{3}\right)^{-1}=\left(\frac{1}{3}\right)^{-1}=\boxed{\text{3 mph.}}$$

3. Determine the exact value of $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\frac{1}{5}}}}$.

   (A) $\dfrac{68}{157}$ **(B)** $\dfrac{77}{157}$ **(C)** $\dfrac{77}{68}$ **(D)** $\dfrac{68}{60}$ **(E)** $\dfrac{77}{60}$

   Answer: $\boxed{\text{A}}$

   Solution: $$\frac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\frac{1}{5}}}}=\frac{1}{2+\dfrac{1}{3+\dfrac{1}{\frac{21}{5}}}}=\frac{1}{2+\dfrac{1}{3+\frac{5}{21}}}=\frac{1}{2+\dfrac{1}{\frac{68}{21}}}=\frac{1}{2+\frac{21}{68}}=\frac{1}{\frac{157}{68}}=\boxed{\frac{68}{157}.}$$

4. A shipping clerk has five boxes of different but unknown weights. Unfortunately, all of the boxes weigh less than 100 pounds and the only scale available only reads weights over 100 pounds. The clerk decides to weigh the boxes in pairs so that each box is weighed with every other box. The weights of all possible pairs are 110, 112, 113, 114, 115, 116, 117, 118, 120, and 121 pounds. Determine the weight of the lightest box.

(A) 51 lbs. **(B)** 52 lbs. **(C)** 53 lbs. **(D)** 54 lbs. **(E)** 55 lbs.

Answer: $\boxed{\text{D}}$

Solution: Given five boxes of weights $a < b < c < d < e$, there are $\binom{5}{2}$=10 ways to choose two of them; the ten weights are shown. Each individual box was paired with all four other boxes. Consequently, if ten equations are written, each of the form $a + b = 110, \ldots, d + e = 121$, and the left-hand and right-hand sides are summed, we obtain

$$4(a + b + c + d + e) = 110 + 112 + 113 + 114 + 115 + 116 + 117 + 118 + 120 + 121 = 1156$$

$$a + b + c + d + e = 289,$$

so all five boxes together weigh 289 pounds.

It is clear that the two lightest boxes together weigh 110 pounds, and the two heaviest boxes together weigh 121 pounds, for a total of 231 pounds. Since all five boxes weigh 289 pounds, the middle box, c, must weigh 58 pounds.

The two lightest boxes together weigh 110 pounds. Which two boxes together weigh 112 pounds? It can only be $a + c$.

So, if $a + c = 112$ and $c = 58$, we can conclude that $a = \boxed{54 \text{ lbs.}}$

An alternate ending to the previous solution: it is clear that the lightest two weights represent $a+b$ and $a + c$, while the heaviest two are $c + e$ and $d + e$. Hence adding the lightest two measurements and the heaviest measurement gives a sum of $(a + b) + (a + c) + (d + e) = a + (a + b + c + d + e)$. Substituting, $110 + 112 + 121 = a + 289$, so $a = 343 - 289 = \boxed{54 \text{ lbs.}}$

Alternate solution: It is possible to work backwards from the answer choices. It is easy to confirm that $b+c$ must be smaller than $b+d$, $b+e$, $c+d$, $c+e$, and $d+e$. Because 115 is the sixth lightest weight, $b + c \leq 115$.

If $a = 51$, then $b = 59$ and $c = 61$, whence $b + c = 120$, too large.

If $a = 52$, then $b = 58$ and $c = 60$, whence $b + c = 118$, too large.

If $a = 53$, then $b = 57$ and $c = 59$, whence $b + c = 116$, too large.

If $a = 55$, then $b = 55$ and $c = 57$. Since $a = b$, this contradicts the statement that all weights are different.

So by elimination, only $\boxed{54 \text{ lbs.}}$ works.

5. If $a \oplus b = ab + a^b$, what is $(1 \oplus 2) \oplus 3$?

(A) 6 **(B)** 14 **(C)** 36 **(D)** 78 **(E)** 93

Answer: $\boxed{\text{C}}$

Solution: $(1 \oplus 2) = 1 \cdot 2 + 1^2 = 3$; $3 \oplus 3 = 3 \cdot 3 + 3^3 = 9 + 27 = \boxed{36.}$

6. Many math leagues in Massachusetts have three questions per round, worth 1, 2, and 3 points, respectively. If a certain student has a 60% chance of answering the one point question correctly, a 40% chance of answering the two point question correctly, and a 20% chance of answering the three point question correctly, then what is the most likely score for the student to earn in the round?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer: $\boxed{\text{B}}$

Solution: Short of calculating all options, two likely "best guesses" present themselves. The first "best guess" is a score of 3, because it can be obtained two different ways ($1 + 2$ or 3). A quick calculation gives the chance of $1 + 2$ to be $0.6 \cdot 0.4 \cdot 0.8 = 19.2\%$ and the chance of 3 to be $0.4 \cdot 0.6 \cdot 0.2 = 4.8\%$, for a total of 24%. The second "best guess" involves the most likely outcomes on each question: question one correct, question two incorrect, question three incorrect. The probability here is $0.6 \cdot 0.6 \cdot 0.8 = 28.8\%$, which is the highest probability that can be obtained, so the answer is $\boxed{1.}$ For completeness, the probabilities are:

Score	Probability
0	19.2%
1	28.8%
2	12.8%
3	24.0%
4	7.2%
5	3.2%
6	4.8%

7. How many positive integers less than 1,000,000 are neither perfect squares nor perfect cubes?

(A) 999,990 **(B)** 999,900 **(C)** 999,000 **(D)** 998,910 **(E)** 998,890

Answer: $\boxed{\text{D}}$

Solution: For convenience, we include 1,000,000 in our calculations, although it is both a perfect square *and* a perfect cube. There are $\sqrt{1000000} = 1000$ perfect squares between 1 and 1,000,000 (namely, 1^2, 2^2, ..., 1000^2). There are $\sqrt[3]{1000000} = 100$ perfect cubes in that range (namely, 1^3, 2^3, ..., 100^3. This leads to an initial estimate of $1,000,000 - 1000 - 100 = 998900$ numbers remaining, but notice that any perfect sixth power between 1 and 1,000,000 has been removed twice, leading to an undercount. There are $\sqrt[6]{1000000} = 10$ perfect sixth powers in the range, so ten must be added to the total to correct the double subtraction. This leads to a final answer of $1000000 - 1000 - 100 + 10 = \boxed{998{,}910.}$

8. How many ordered pairs (a, b) of positive integers exist such that $\frac{1}{a} + \frac{5}{b} = \frac{1}{2}$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer: $\boxed{\text{C}}$

Solution: Multiplying both sides of the equation by $2ab$, we have

$$2b + 10a = ab$$

$$\begin{aligned}
2b &= ab - 10a \\
2b &= a(b-10) \\
a &= \frac{2b}{b-10} \\
a &= \frac{2b - 20 + 20}{b-10} \\
a &= 2 + \frac{20}{b-10}.
\end{aligned}$$

Therefore we know that, because a is an integer, $\frac{20}{b-10}$ is. So $b-10$ is a factor of 20:

$$(b-10) \in \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20\}$$

$$b \in \{10 \pm 1, 10 \pm 2, 10 \pm 4, 10 \pm 5, 10 \pm 10, 10 \pm 20\}$$

$$b \in \{-10, 0, 5, 6, 8, 9, 11, 12, 14, 15, 20, 30\}$$

The first two can be excluded (not positive integers). Using $a = \frac{2b}{b-10}$, this leads to the ten ordered pairs

$$(-2,5) \quad (-3,6) \quad (-8,8) \quad (-18,9) \quad (22,11) \quad (12,12) \quad (7,14) \quad (6,15) \quad (4,20) \quad (3,30)$$

The first four of these are not both positive integers, but the remaining $\boxed{\text{six}}$ are.

9. Every Martian is one of two types: 10% of Martians are of type P, while the other 90% are of type NP. If two randomly-selected Martians are in the same crater, what is the probability that they are of the same type?

   (A) 1% **(B)** 9% **(C)** 18% **(D)** 81% **(E)** 82%

   Answer: $\boxed{\boxed{\text{E}}}$

   Solution: The probability that both Martians are of type NP is $90\% \cdot 90\% = 81\%$; the probability that both Martians are of type P is $10\% \cdot 10\% = 1\%$. So the total probability is $81\% + 1\% = \boxed{82\%.}$

10. The sum of a certain infinite geometric series is 20. When all the terms in the series are squared, the sum of the resulting series is 80. If the first term of the original series is expressed in lowest terms as $\frac{p}{q}$, find $p + q$.

    (A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

    Answer: $\boxed{\boxed{\text{C}}}$

    Solution: Using the formula for the sum of an infinite geometric series, the first two sentences state that

    $$\frac{a}{1-r} = 20 \qquad \text{and} \qquad \frac{a^2}{1-r^2} = 80.$$

    (If the terms are squared, the common ratio is squared.) Because the second equation factors as

    $$\frac{a}{1-r} \cdot \frac{a}{1+r} = 80,$$

we can divide the first equation into this to deduce that

$$\frac{a}{1+r} = 4.$$

Using the first equation and this new equation, we have

$$\frac{a}{1-r} = 20 \qquad \text{and} \qquad \frac{a}{1+r} = 4$$

$$a = 20 - 20r \qquad\qquad a = 4 + 4r$$

$$20 - 20r = 4 + 4r$$

$$16 = 24r$$

$$r = \frac{2}{3}.$$

Plugging into $a = 4 + 4r$, we deduce that $a = 4 + \frac{8}{3} = \frac{20}{3}$. So the sum of the numerator and denominator of the first term (a) is $20 + 3 = \boxed{23.}$

11. Find the distance between the circle $(x-5)^2 + (y+3)^2 = 1$ and the line $5x + 12y - 4 = 0$.

(A) $\frac{1}{13}$ **(B)** $\frac{2}{13}$ **(C)** $\frac{3}{13}$ **(D)** $\frac{3}{15}$ **(E)** $\frac{4}{15}$

Answer: $\boxed{\boxed{\text{B}}}$

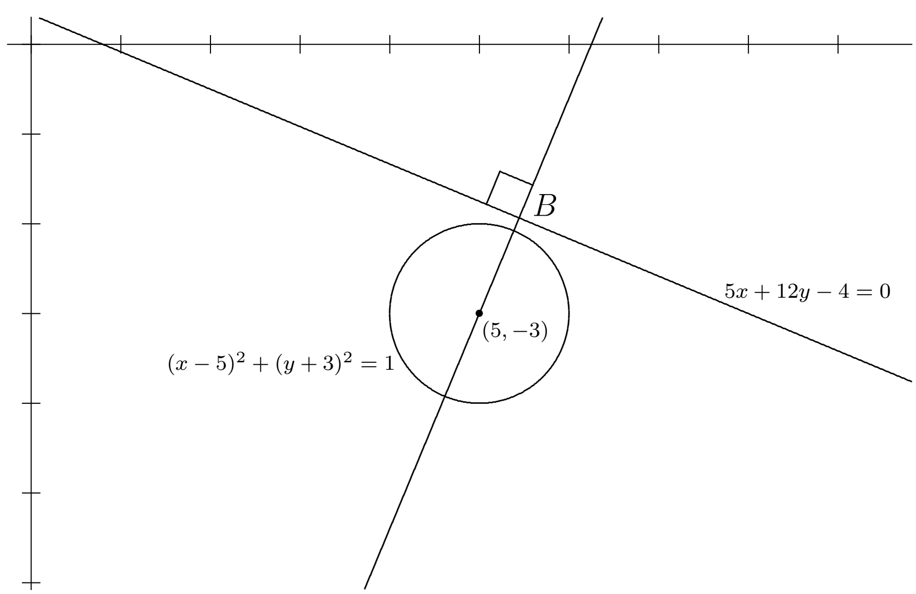
Solution: The distance from a point (x_0, y_0) to a line $Ax + By + C = 0$ is

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

The center of the circle is $(5, -3)$, so the distance from this point to the line $5x + 12y - 4 = 0$ is

$$\frac{5 \cdot 5 + 12 \cdot (-3) - 4}{\sqrt{5^2 + 12^2}} = \frac{|25 - 36 - 4|}{\sqrt{169}} = \frac{15}{13}.$$

From this, subtract the radius of the circle, one, to obtain the answer of $\boxed{2/13.}$



An inefficient solution method is to consider the line through $(5, -3)$ which is perpendicular to the given line $5x + 12y - 4 = 0$. Because the given line has slope $-\frac{5}{12}$, the perpendicular line has slope $\frac{12}{5}$, leading to the line $y+3 = \frac{12}{5}(x-5)$. This intersects the original line at $(\frac{920}{169}, -\frac{327}{169})$; the distance between this point and $(5, -3)$ is $\frac{15}{13}$; then subtract one to obtain $\boxed{2/13.}$

A third, also inefficient, solution involves calculus. The distance from the point to the line is a minimization problem: because an equivalent equation of the line is $y = \frac{1}{3} - \frac{5}{12}x$, each point on the line is of the form $(x, \frac{1}{3} - \frac{5}{12}x)$. The square of the distance from any such point to $(5, -3)$ is $(x-5)^2 + (\frac{1}{3} - \frac{5}{12}x + 3)^2$. Taking the derivative, we have

$$2(x-5) + 2\left(\frac{10}{3} - \frac{5}{12}x\right)\left(-\frac{5}{12}\right) = 0$$

$$(x-5) + \left(\frac{10}{3} - \frac{5}{12}x\right)\left(-\frac{5}{12}\right) = 0$$

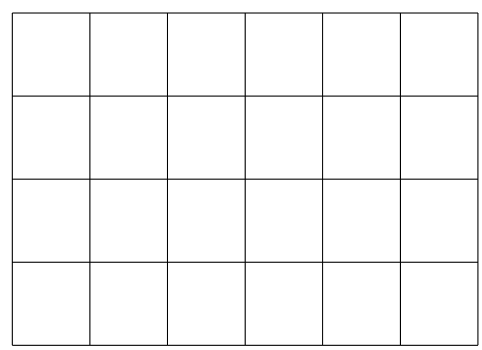
$$x - 5 - \frac{50}{36} + \frac{25}{144}x = 0$$

$$\frac{169}{144}x = \frac{230}{36}$$

$$x = \frac{920}{169}$$

Then the y-coordinate must be obtained; the distance to $(5, -3)$ must be calculated, and one unit must be subtracted, leading to the same answer.

12. How many rectangles *that are not squares* are in this grid? For example, the entire 4×6 grid is a rectangle but not a square; any $n \times n$ rectangle is a square and should not be counted.



(A) 160 **(B)** 170 **(C)** 185 **(D)** 200

(E) 210

Answer: $\boxed{\text{A}}$

Solution: Although it would be possible to count the number of rectangles of each size (the number of $w \times h$ rectangles is $(7-w)(5-h)$ and there are "only" 20 sizes to consider, it is far easier to use basic combinatorics. Temporarily allowing squares, a rectangle can be described by its four sides — two horizontal and two vertical. Because there are five horizontal lines to choose from and seven vertical lines to choose from, the number of rectangles, *including squares,* is $\binom{5}{2}\binom{7}{2} = \frac{5\cdot4}{2\cdot1} \cdot \frac{7\cdot6}{2\cdot1} = 10 \cdot 21 = 210$.

To exclude squares, we use the $(7-w)(5-h)$ method mentioned above. The number of 1×1, 2×2, 3×3, and 4×4 squares are $6 \cdot 4 = 24$, $5 \cdot 3 = 15$, $4 \cdot 2 = 8$, and $3 \cdot 1 = 3$, respectively, for a total of $24 + 15 + 8 + 3 = 50$ squares.

Thus the answer to the question is $210 - 50 = \boxed{160.}$

13. If $a = \log 12$, $b = \log 21$, $c = \log 11$, and $d = \log 22$, then $\log\left(\frac{1}{7}\right)$ can be expressed in the form $P(a-b)+Q(c-d)$, where P and Q are integers. Find $7P+Q$.

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

Answer: $\boxed{\text{A}}$

Solution: We have $\log 2 = \log \frac{22}{11} = d-c$; then $\log 12 = 2\log 2 + \log 3$, which gives $\log 3 = a-2(d-c) = a+2c-2d$. Then $\log 21 = \log 3 + \log 7$, which gives $\log 7 = b-(a+2c-2d) = -a+b-2c+2d$. Since $\log\frac{1}{7} = -\log 7$, we have $\log\frac{1}{7} = a-b+2c-2d = 1(a-b)+2(c-d)$. Therefore the answer is $7\cdot 1+2 = \boxed{9.}$

14. Given that $\cos x + \cos y + \cos z = \sin x + \sin y + \sin z = 0$, find the numerical value of

$$\cos(x-y)+\cos(y-z)+\cos(z-x).$$

(A) $-\dfrac{3}{2}$ **(B)** $-\dfrac{1}{2}$ **(C)** 0 **(D)** $\dfrac{1}{2}$ **(E)** $\dfrac{\sqrt{3}}{2}$

Answer: $\boxed{\text{A}}$

Solution: Consider the equations $\cos x+\cos y+\cos z = 0$ and $\sin x+\sin y+\sin z = 0$. Squaring both, we have

$$\cos^2 x+\cos^2 y+\cos^2 z+2\cos x\cos y+2\cos x\cos z+2\cos y\cos z = 0$$

$$\sin^2 x+\sin^2 y+\sin^2 z+2\sin x\sin y+2\sin x\sin z+2\sin y\sin z = 0$$

Adding these, and using the Pythagorean identity, we have

$$1+1+1+2(\cos x\cos y+\sin x\sin y+\cos x\cos z+\sin x\sin z+\cos y\cos z+\sin y\sin z) = 0.$$

Because $\cos(a-b) = \cos(b-a) = \cos a\cos b+\sin a\sin b$, this simplifies to

$$3+2\left(\cos(x-y)+\cos(z-x)+\cos(y-z)\right) = 0,$$

so

$$\cos(x-y)+\cos(y-z)+\cos(z-x) = \boxed{-\tfrac{3}{2}}.$$

15. Two brothers sold a flock of sheep that they owned. The number of dollars they received for each sheep was equal to the number of sheep in the herd. The buyer of the flock paid the brothers by giving the older brother \$10, then the younger brother \$10, then the older brother \$10, and so on, until all of the money was gone. The last brother to receive money was the younger brother, who received less than \$10, so the older brother gave him some money to make their amounts equal. How much money was given?

(A) \$1 **(B)** \$2 **(C)** \$3 **(D)** \$4 **(E)** \$5

Answer: $\boxed{\text{B}}$

Solution: Because "the number of dollars they received for each sheep was equal to the number of sheep in the herd," the amount received was a perfect square. Because "the last brother to receive money was the younger brother, who received less than \$10," we know that the amount received was not a multiple of ten, and moreover that it contained an odd number of tens, plus an amount between \$1 and \$9.

The first twenty perfect squares are 1, 4, 9, $\underline{16}$, 25, $\underline{36}$, 49, 64, 81, 100, 121, 144, 169, $\underline{196}$, 225, $\underline{256}$, 289, 324, 361, and 400; the only squares that satisfy the conditions are the underlined four (16, 36, 196, 256), all of which end in six. Because the older brother received \$10 before the younger received \$6, $\boxed{\$2}$ must be given (as if both brothers received \$8 in the last step).

Of course this is not rigorous. More rigorously, we are squaring a number $10t + r$, where t is the tens of sheep sold, and r is the remainder of sheep (fewer than ten). The payment was $(10t+r)^2 = 100t^2+20tr+r^2$. But the payment contained an odd number of tens, and $100t^2+20tr = 20(5t^2+tr)$ is clearly divisible by 20 (an even number of tens). So for $100t^2 + 20tr + r^2$ to have an odd number of tens, then, r^2 itself must contain an odd number of tens; we simply check the squares:

n	n^2
0	0
1	1
2	4
3	9
4	$\underline{16}$

n	n^2
5	25
6	$\underline{36}$
7	49
8	64
9	81

Only 4^2 and 6^2 give an even number of tens; so we can deduce that $r = 4$ or $r = 6$. This means that the amount given to the brothers ended in six. Hence the older brother had four additional dollars; hence the amount given was half of that, or $\boxed{\$2.}$

16. There are infinitely many sequences whose first four terms are 2, 0, 0, 7. If f is the polynomial function of smallest degree such that $f(1) = 2$, $f(2) = f(3) = 0$, and $f(4) = 7$, find $f(5)$.

(A) 15 **(B)** 20 **(C)** 21 **(D)** 24 **(E)** 26

Answer: $\boxed{\boxed{\text{E}}}$

Solution: Using Newton's forward difference formula (often called the "method of finite differences"), it can be quickly determined that the simplest polynomial equation is $f(n) = 1\binom{n}{0} + 1\binom{n}{1} - 3\binom{n}{2} + 5\binom{n}{3}$. The coefficients preceding the binomial coefficients come from a difference table:

$n =$	0		1		2		3		4
$f(n) =$	$\boxed{1}$		**2**		**0**		**0**		**7**
		$\boxed{1}$		-2		0		7	
			$\boxed{-3}$		2		7		
				$\boxed{5}$		5			

where the boldfaced **2 0 0 7** are the four given values. This simplifies to $f(n) = \frac{1}{6}\left(5n^3 - 24n^2 + 25n + 6\right)$ Plugging in $n = 5$ gives $f(5) = \boxed{26.}$

However, none of this simplification is necessary, because the method of finite differences also predicts future values. Simply extend the difference table to the right:

$n=$	1		2		3		4		5
$f(n)=$	**2**		**0**		**0**		**7**		$\boxed{26}$
		-2		0		7		19	
			2		7		12		
				5		5			

and the answer, $\boxed{26,}$ is shown.

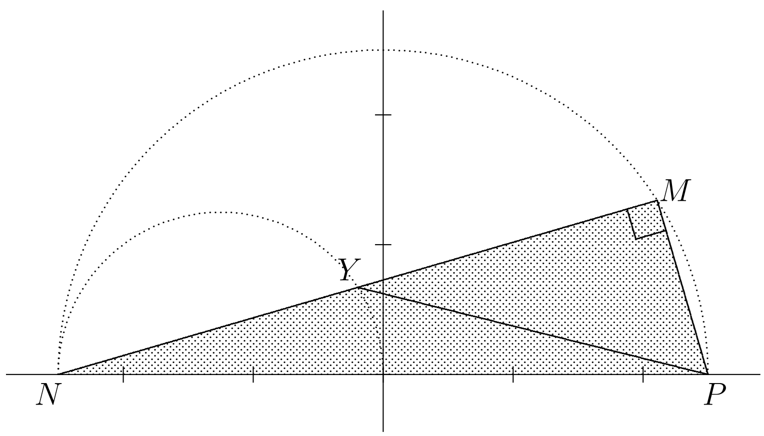
Of course it is also possible to set up a system of four simultaneous equations by plugging $(1,2)$, $(2,0)$, $(3,0)$, and $(4,7)$ into a generic cubic $y = ax^3 + bx^2 + cx + d$ and solving for a, b, c, and d, but this is a lot of unnecessary work.

17. In right triangle MNP, Y is the midpoint of $\overline{MN}$. If hypotenuse $\overline{NP}$ has length 17, then $\overline{YP}$ cannot be shorter than p or greater than q. Find $p+q$.

(A) 24 **(B)** 25 **(C)** 25.5 **(D)** $17+\dfrac{\sqrt{10}}{4}$ **(E)** $17+\dfrac{17\sqrt{10}}{4}$

Answer: $\boxed{\boxed{\text{C}}}$

Solution: $\overline{YP}$, the median, will approach length 17 if MN is very small; and it will approach length 8.5 if MP is very small. So the answer is $8.5+17=\boxed{25.5.}$



Notice that, given a fixed $\overline{NP}$, the right angle at M restricts M's placement to a circle; and because Y is always half the distance from N to M, Y is restricted to a circle (which is a dilation of M's circle, the dilation centered at N).

18. Emma has ten apples, and wants to give them to (up to) three friends. How many different ways can she do so? (Assume that the apples are all identical, but her friends are not; it is possible, for example, for her to give one friend all ten apples.)

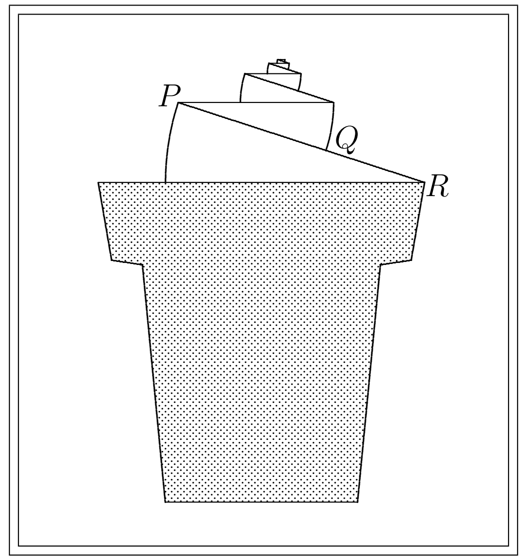
(A) 36 **(B)** 63 **(C)** 66 **(D)** 1000 **(E)** 3^{10}

Answer: $\boxed{\boxed{\text{C}}}$

Solution: Pretend all ten apples are red, and temporarily introduce two identical yellow apples. There are $\binom{12}{2}$ ways that the twelve apples (ten identical red apples, two identical yellow apples) can be arranged in a row: by choosing which two places in the row will contain the yellow apples, the entire arrangement is determined. But once the apples are placed in a row in this way, the first person can take all red apples until the first yellow apple; the second person can take all red apples between the two yellow apples; and the third person can take all red apples after the second yellow apple. Each distribution of apples can be shown as a row of twelve apples, and each row of twelve apples is exactly one distribution. Therefore there are $\binom{12}{2} = \boxed{66}$ ways to distribute the apples.

Note that if a yellow apple is first, the first person gets no apples; if a yellow apple is last, the third person gets no apples; and if the two yellow apples are adjacent, the second person gets no apples.

19. A soft-serve ice cream cone is shown in a photograph. Each three-dimensional swirl of ice cream (visible as a sector in the photograph) is similar to every other; in fact, any two neighboring swirls together are similar to any other two neighboring swirls together, and the swirls continue upward forever. If the volume of the bottom swirl is $\mathcal{S}$, the volume of ice cream in the cone is $\mathcal{C}$, and $QR = \frac{2}{5}PR$, find the total volume of ice cream.



(A) $\mathcal{C} + \dfrac{25}{21}\mathcal{S}$ **(B)** $\mathcal{C} + \dfrac{125}{98}\mathcal{S}$

(C) $\mathcal{C} + \dfrac{25}{16}\mathcal{S}$ **(D)** $\mathcal{C} + \dfrac{5}{3}\mathcal{S}$

(E) $\mathcal{C} + \dfrac{5}{2}\mathcal{S}$

Answer: $\boxed{\text{B}}$

Solution: Because $\overline{PQ}$ has 3/5 the length of $\overline{PR}$, a corresponding length (because the bottom line of the bottom swirl has the same length as $\overline{PR}$), we have that the volume of the second swirl is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$ the volume of the first swirl. And, because each consecutive pair of swirls is similar to every other consecutive pair, this ratio is maintained as the swirls go upward. Hence the total volume is

$$\mathcal{C} + \frac{\mathcal{S}}{1 - \frac{27}{125}}$$

$$\mathcal{C} + \frac{\mathcal{S}}{98/125}$$

$$\boxed{\mathcal{C} + \frac{125}{98}\mathcal{S}.}$$

20. For any number a, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a, and let $\{a\}$ denote the fractional part of a; in other words, $\{a\} = a - \lfloor a \rfloor$. For example, $\lfloor 5.3 \rfloor = 5$ and $\{5.3\} = 0.3$. Given that

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 4.2 \\ y + \lfloor z \rfloor + \{x\} &= 5.8 \\ z + \lfloor x \rfloor + \{y\} &= 7.4, \end{aligned}$$

find $2x + 3y + z$.

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Answer: $\boxed{\text{E}}$

Solution: Adding all three equations together, one gets

$$x + \lfloor x \rfloor + \{x\} + y + \lfloor y \rfloor + \{y\} + z + \lfloor z \rfloor + \{z\} = 17.4$$

Since the integral part of a number plus its fractional part is the number itself, this simplifies to

$$2x + 2y + 2z = 17.4$$

$$x + y + z = 8.7$$

Similarly, adding two equations at a time gives

Eqs I & II	Eqs II & III	Eqs I & III
$x + y + z + \lfloor y \rfloor + \{x\} = 10$	$x + y + z + \lfloor z \rfloor + \{y\} = 13.2$	$x + y + z + \lfloor x \rfloor + \{z\} = 11.6$
$\lfloor y \rfloor + \{x\} = 1.3$	$\lfloor z \rfloor + \{y\} = 4.5$	$\lfloor x \rfloor + \{z\} = 2.9$
$\lfloor y \rfloor = 1.0 \qquad \{x\} = 0.3$	$\lfloor z \rfloor = 4.0 \qquad \{y\} = 0.5$	$\lfloor x \rfloor = 2.0 \qquad \{z\} = 0.9$

Combining, we deduce that $x = 2.3$, $y = 1.5$, and $z = 4.9$. Therefore $2x + 3y + z = 4.6 + 4.5 + 4.9 = \boxed{14.}$

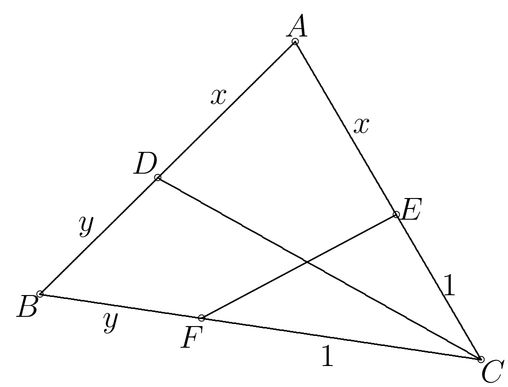
21. In triangle ABC, D lies on $\overline{AB}$, E lies on $\overline{AC}$, and F lies on $\overline{BC}$. The segments shown have lengths in the following ratios:

$$AE : EC = x : 1$$

$$BF : FC = y : 1$$

$$AD : DB = x : y$$

What must be true about x and y in order that $\overline{EF}$ bisects $\overline{CD}$?



(A) $x = y$ **(B)** $x^2 = y^2$ **(C)** $xy = 1$

(D) $x + y = 2$ **(E)** $2xy = x + y$

Answer: $\boxed{\text{E}}$

Solution: The method of mass points uses two facts from physics:

(a) To balance two objects on a seesaw, an object that is n times as heavy should be $1/n$ times as far from the fulcrum; or an object that is $1/n$ times as heavy should be n times as far from the fulcrum. Specifically, the product of the mass and distance from the fulcrum must be the same on both sides of the fulcrum.

(b) To balance a two-dimensional object on a point, the object must be balanced across two lines that intersect at that point.

Using this principle, we will balance the triangle on the intersection of $\overline{CD}$ and $\overline{EF}$.

First, we balance across $\overline{CD}$. If we place a mass x on B, and a mass y on A, the system balances across $\overline{CD}$. (We assume the triangle itself has no mass.)

Next, we balance across $\overline{EF}$. To counteract the mass at B, we need a mass xy on C. But to counteract the mass at A, we also need a mass xy on C. Consequently, to balance the system, we need a weight $2xy$ on C. With these three weights in place, the triangle will balance on the intersection of $\overline{CD}$ and $\overline{EF}$.

A final principle of mass points states that two masses in balance can be moved to their fulcrum without affecting the system. So we can move the masses at A and B to D without disturbing the system.

This leaves a mass $x + y$ at D and a mass $2xy$ at C. Because $\overline{CD}$ is bisected by the balancing point, these masses are at equal distances from the fulcrum, which means the masses are equal: $\boxed{2xy = x + y.}$

Alternate solution: Suggested by Feiqi Jiang.

Draw $\triangle DEF$. Notice that $[DEF] = [CEF]$, because both triangles have equal bases and altitudes.

Because

$$[ADE] + [BDF] + [CEF] + [DEF] = [ABC]$$
$$[ADE] + [BDF] + 2[CEF] = [ABC],$$

we have

$$\frac{[ADE]}{[ABC]} + \frac{[BDF]}{[ABC]} + 2\frac{[CEF]}{[ABC]} = 1.$$

Therefore the problem can be solved if the three ratios above are determined in terms of x and y. Using the formula $A = \frac{1}{2}ab\sin\theta$, we have

$$\frac{[ADE]}{[ABC]} = \frac{\frac{1}{2}x^2\sin(\angle DAE)}{\frac{1}{2}(x+y)(x+1)\sin(\angle DAE)} = \frac{x^2}{(x+y)(x+1).}$$

The other two ratios can be computed similarly, giving

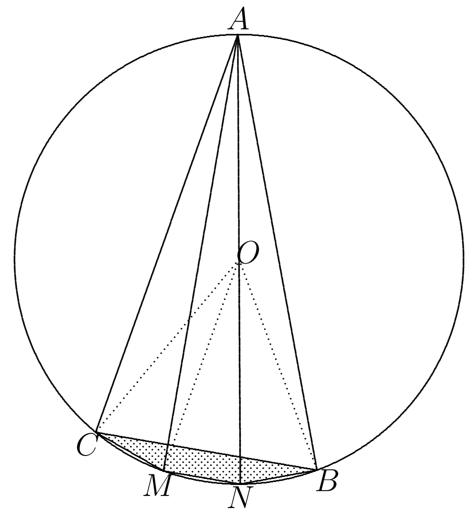
$$\frac{x^2}{(x+y)(x+1)} + \frac{y^2}{(x+y)(y+1)} + 2\frac{1}{(y+1)(x+1)} = 1$$
$$x^2(y+1) + y^2(x+1) + 2(x+y) = (x+y)(x+1)(y+1)$$
$$x^2y + x^2 + xy^2 + y^2 + 2x + 2y = x^2y + xy^2 + x^2 + y^2 + 2xy + x + y$$
$$2x + 2y = 2xy + x + y$$
$$\boxed{x + y = 2xy.}$$

22. Triangle ABC is inscribed in circle O of radius one. Angle A measures $30°$ and angle B measures $70°$. M and N are points on circle O such that M is closer to C than to B and $\overline{AM}$ and $\overline{AN}$ trisect $\angle A$. The area of quadrilateral $BNMC$ is $P\sin 20° - \sqrt{Q}$, where P and Q are positive rational numbers in simplest form. Find $16(P+Q)$.

(A) 21 **(B)** 23 **(C)** 25 **(D)** 27 **(E)** 29

Answer: $\boxed{\text{D}}$

Solution: Draw radii $\overline{OC}$, $\overline{OM}$, $\overline{ON}$, and $\overline{OB}$. All have length one unit.



Because $\angle A$ was trisected, $\angle CAM = \angle MAN = \angle NAB = 10°$. By the Inscribed Angle Theorem, therefore, $\widehat{CM} = \widehat{MN} = \widehat{NB} = 20°$. To find the area of quadrilateral $BNMC$, we compute

$$\alpha\triangle COM + \alpha\triangle MON + \alpha\triangle NOB - \alpha\triangle COB$$

$$3 \cdot \frac{1}{2}(1)(1)\sin 20° - \frac{1}{2}(1)(1)\sin 60°$$

$$\frac{3}{2}\sin 20° - \frac{\sqrt{3}}{4}$$

(We have used the area formula $\frac{1}{2}ab\sin\theta$, where a and b are the sides of a triangle and θ is the measure of the included angle.) Consequently $P = \frac{3}{2}$ and $Q = \frac{3}{16}$, so $16P + 16Q = 24 + 3 = \boxed{27.}$

23. A dimwitted painter is hired to paint a 200-foot line in the center of a straight road. His 200-step procedure is as follows:

    1. He fills his brush, paints one foot of the line, then walks one foot back to the paint can.
    2. He fills his brush, walks one foot to the end of the painted line, paints another foot, and walks two feet back to the paint can.
    3. He fills his brush, walks two feet to the end of the painted line, paints another foot, and walks three feet back to the paint can.

    (etc.)

    The painter never moves the can!

    In the first hour, the painter is able to complete 100 of these steps (painting 100 feet of the line). In the second hour, because of the additional walking distance, he is only able to complete 50 more steps (painting 50 more feet of the line).

    Assuming his rates of walking and painting are constant, that filling the brush takes no time, and that he does not rest or pause, how long will it take to complete all 200 steps?

(A) between 3 and 3.5 hours **(B)** between 3.5 and 4 hours **(C)** between 4 and 4.5 hours

(D) between 4.5 and 5 hours **(E)** more than 5 hours

Answer: A

Solution: During the first hour, the painter walks $0+1+2+\cdots+99 = \frac{99\cdot100}{2} = 4950$ feet *to* the line, paints 100 feet *at* the line, and walks $1+2+3+\cdots+100 = \frac{101\cdot100}{2} = 5050$ feet *from* the line. So he can walk 10000 feet and paint 100 feet in one hour.

During the second hour, the painter walks $100+101+102+\cdots+149 = \frac{249\cdot50}{2} = 6225$ feet *to* the line, paints 50 feet *at* the line, and walks $101+102+103+\cdots+150 = \frac{251\cdot50}{2} = 6275$ feet *from* the line. So he can also walk 12500 feet and paint 50 feet in one hour.

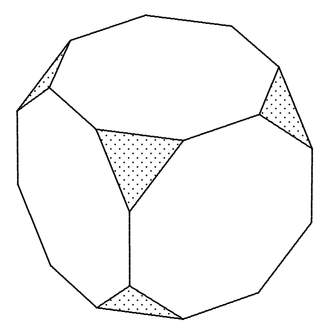
Subtracting, we see that the painter walked an additional 2500 feet in the second hour, while painting 50 fewer feet. So 2500 feet of walking equals 50 feet of painting, so 50 feet of walking equal one foot of painting. This means that, since the painter walked 10000 feet and painted 100 feet in the first hour, he could walk 15000 feet in an hour.

To finish the job, the painter must walk $150+151+152+\cdots+199 = \frac{349\cdot50}{2} = 8725$ feet *to* the line, paint 50 feet *at* the line, and walk $151+152+153+\cdots+200 = \frac{351\cdot50}{2} = 8775$ feet *from* the line. So he must walk 17500 feet and paint 50 feet to finish the job.

From his second hour, we know he can walk 12500 feet and paint 50 feet in an hour; so to finish the job he will require one hour (to do these things) plus enough time to walk 5000 feet extra. Since the painter can walk 15000 feet in an hour, the additional 5000 feet will take him 1/3 hour, or 20 minutes.

Therefore the complete job will take $1+1+\frac{4}{3} = \frac{10}{3}$ hours, which is between 3 and 3.5 hours.

24. The corners of a cube are sliced off, resulting in the solid shown to the right. If all edges of the solid have length 1, find the volume of the solid.



(A) $\dfrac{80}{3}$ **(B)** $27-\dfrac{\sqrt{2}}{3}$ **(C)** $\dfrac{20}{3}+5\sqrt{2}$

(D) $7+\dfrac{14}{3}\sqrt{2}$ **(E)** $7+5\sqrt{2}-\dfrac{\sqrt{3}}{3}$

Answer: D

Solution: The volume is that of a cube with eight corners (equilateral triangular pyramids) cut off.

To find the volume of the cube, we need the edge length. This is fairly routine: each face is a octagon with side length 1, so the missing corners are legs of 45-45-90 right triangles. Since the hypotenuses of those triangles have length 1, the legs have lengths $\frac{\sqrt{2}}{2}$. Hence the side length of the cube is $1+2\cdot\frac{\sqrt{2}}{2} = 1+\sqrt{2}$. This gives the original cube a volume of $(1+\sqrt{2})^3 = 7+5\sqrt{2}$

To find the volume of each removed corner, we need the base and the height. The base, being an equilateral triangle, has area $\frac{\sqrt{3}}{4}s^2$, and $s=1$, so the area is $\frac{\sqrt{3}}{4}$. The height must be deduced using a right triangle, either using the slant height as the hypotenuse and the smaller piece of the median as a leg; or using a lateral edge of the pyramid as the hypotenuse and the larger piece of the median as a leg. Either way, it is necessary to know that the medians are divided into a 2 : 1 ratio by the other medians.

Using the former method, we have

$$\begin{aligned} h^2 + \left(\frac{1}{3} \cdot \frac{\sqrt{3}}{2}\right)^2 &= \left(\frac{1}{2}\right)^2 \\ h^2 + \left(\frac{1}{2\sqrt{3}}\right)^2 &= \left(\frac{1}{2}\right)^2 \\ h^2 = \frac{1}{4} - \frac{1}{12} &= \frac{1}{6}. \end{aligned}$$

Using the latter method, we have

$$\begin{aligned} h^2 + \left(\frac{2}{3} \cdot \frac{\sqrt{3}}{2}\right)^2 &= \left(\frac{\sqrt{2}}{2}\right)^2 \\ h^2 = \frac{1}{2} - \frac{1}{3} &= \frac{1}{6}. \end{aligned}$$

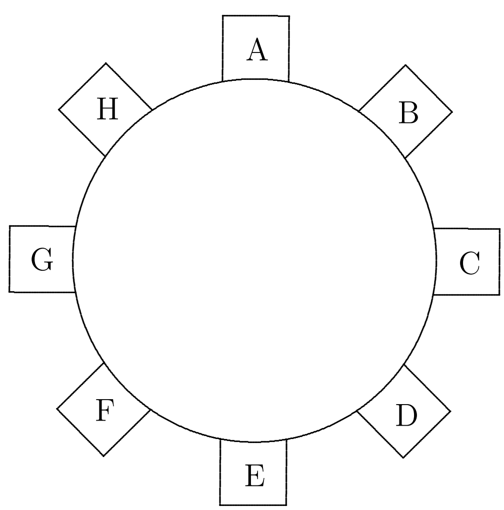
Either way, $h = \frac{\sqrt{6}}{6}$, so the volume of each removed corner was

$$\frac{1}{3}Bh = \frac{1}{3}\left(\frac{\sqrt{3}}{4}\right)\frac{\sqrt{6}}{6} = \frac{3\sqrt{2}}{72} = \frac{\sqrt{2}}{24}.$$

Because there are eight of them, the total removed volume was $\frac{\sqrt{2}}{3}$. Hence the volume of the truncated cube is $(1+\sqrt{2})^3 = 7 + 5\sqrt{2} - \frac{\sqrt{2}}{3} = \boxed{7 + \frac{14}{3}\sqrt{2}.}$

25. Chan, Daniel, and six other math students are sitting at a round table, each one thoroughly engrossed in a different math topic. The table has eight seats, labeled "A" through "H."

Baiyang
Chan
Christine
Daniel
Haisha
Michael
Shuyi
Tom



Algebra
Calculus
Combinatorics
Geometry
Graph Theory
Logic
Number Theory
Trigonometry

(a) Michael is directly opposite Haisha and to the immediate left of Shuyi, who is directly opposite the student who is working on combinatorics, who, in turn, is on the immediate right of the student who is studying number theory.

(b) The student solving a logic problem is in chair F. Tom is in chair G, directly opposite the student who is working on algebra (who is not Baiyang).

(c) The student studying graph theory is on the immediate right of the one working on a geometry problem, who is not sitting in chair E.

(d) Christine, who is not in chair D, is not the one who is studying trigonometry.

(e) The student in chair A is working on calculus.

(f) No student's name has the same initial as the chair in which he or she is sitting.

Determine which lettered seat is being used by Daniel.

(A) Chair A **(B)** Chair B **(C)** Chair C **(D)** Chair E **(E)** Chair H

Answer: C

Solution: First fill in the diagram with known information: (b) says that the logic problem is in F, Tom is in G, and algebra is in C; (e) says that calculus is in A.

Seat	**Name**	**Subject**
A		**Calculus**
B		
C		**Algebra**
D		
E		
F		**Logic**
G	**Tom**	
H		

By (c), we know that geometry and graph theory are adjacent. There are only two places there they could now fit: E and D, or H and G. But (c) excludes E and D, so geometry is in H and graph theory is in G (studied by Tom).

Seat	**Name**	**Subject**
A		Calculus
B		
C		Algebra
D		
E		
F		Logic
G	Tom	**Graph Theory**
H		**Geometry**

By (a), we know number theory and combinatorics are adjacent. Now there is only one place for these: E and D. So number theory is in E, and combinatorics is in D. By elimination, trigonometry (the last remaining subject) is in B. Also using (a), this means Michael is in A (studying calculus), Shuyi is in H (studying geometry), and Haisha is in E (studying number theory).

Seat	**Name**	**Subject**
A	**Michael**	Calculus
B		**Trigonometry**
C		Algebra
D		**Combinatorics**
E	**Haisha**	**Number Theory**
F		Logic
G	Tom	Graph Theory
H	**Shuyi**	Geometry

By (d), Christine is not in D; because she is not studying trigonometry, she is not in B; and by (f) she is not in C. Only seat F remains, so Christine is in F.

Seat	**Name**	**Subject**
A	Michael	Calculus
B		Trigonometry
C		Algebra
D		Combinatorics
E	Haisha	Number Theory
F	**Christine**	Logic
G	Tom	Graph Theory
H	Shuyi	Geometry

By (b), Baiyang is not in C, because she is not studying algebra; by (f) she is not in B. Only seat D remains, so Baiyang is in D.

Seat	**Name**	**Subject**
A	Michael	Calculus
B		Trigonometry
C		Algebra
D	**Baiyang**	Combinatorics
E	Haisha	Number Theory
F	Christine	Logic
G	Tom	Graph Theory
H	Shuyi	Geometry

Two students remain: Chan and Daniel. By (f) Chan cannot be in C, so he must be in B. This means Daniel is in $\boxed{\text{Chair C.}}$

Seat	**Name**	**Subject**
A	Michael	Calculus
B	**Chan**	Trigonometry
C	**Daniel**	Algebra
D	Baiyang	Combinatorics
E	Haisha	Number Theory
F	Christine	Logic
G	Tom	Graph Theory
H	Shuyi	Geometry

The 2007 Massachusetts Mathematics Olympiad and these solutions were both typeset in Computer Modern using the mathematical typesetting package LaTeX.

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