Forty-Fourth Annual

MASSACHUSETTS MATHEMATICS OLYMPIAD

2007–2008

Conducted by

The Massachusetts Association of Mathematics Leagues

Sponsored by

The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

Tuesday, October 23, 2007

1. Allison draws five squares and eight triangles. How many sides do these shapes have altogether?

   (A) 13 **(B)** 26 **(C)** 39 **(D)** 44 **(E)** 47

2. On my walk to school, I traveled at three different speeds: for the first mile, I traveled at 6 mph; for the second mile, I traveled at 2 mph; and for the final mile, I traveled at 3 mph. What was my average speed for the entire trip?

   (A) 3 mph **(B)** 3.5 mph **(C)** $3.\overline{6}$ mph **(D)** 4.25 mph **(E)** 5.5 mph

3. Determine the exact value of $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\frac{1}{5}}}}$.

   (A) $\dfrac{68}{157}$ **(B)** $\dfrac{77}{157}$ **(C)** $\dfrac{77}{68}$ **(D)** $\dfrac{68}{60}$ **(E)** $\dfrac{77}{60}$

4. A shipping clerk has five boxes of different but unknown weights. Unfortunately, all of the boxes weigh less than 100 pounds and the only scale available only reads weights over 100 pounds. The clerk decides to weigh the boxes in pairs so that each box is weighed with every other box. The weights of all possible pairs are 110, 112, 113, 114, 115, 116, 117, 118, 120, and 121 pounds. Determine the weight of the lightest box.

   (A) 51 lbs. **(B)** 52 lbs. **(C)** 53 lbs. **(D)** 54 lbs. **(E)** 55 lbs.

5. If $a \oplus b = ab + a^b$, what is $(1 \oplus 2) \oplus 3$?

   (A) 6 **(B)** 14 **(C)** 36 **(D)** 78 **(E)** 93

6. Many math leagues in Massachusetts have three questions per round, worth 1, 2, and 3 points, respectively. If a certain student has a 60% chance of answering the one point question correctly, a 40% chance of answering the two point question correctly, and a 20% chance of answering the three point question correctly, then what is the most likely score for the student to earn in the round?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

7. How many positive integers less than 1,000,000 are neither perfect squares nor perfect cubes?

(A) $999,990$ **(B)** $999,900$ **(C)** $999,000$ **(D)** $998,910$ **(E)** $998,890$

8. How many ordered pairs (a, b) of positive integers exist such that $\frac{1}{a} + \frac{5}{b} = \frac{1}{2}$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

9. Every Martian is one of two types: 10% of Martians are of type P, while the other 90% are of type NP. If two randomly-selected Martians are in the same crater, what is the probability that they are of the same type?

(A) 1% **(B)** 9% **(C)** 18% **(D)** 81% **(E)** 82%

10. The sum of a certain infinite geometric series is 20. When all the terms in the series are squared, the sum of the resulting series is 80. If the first term of the original series is expressed in lowest terms as $\frac{p}{q}$, find $p + q$.

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

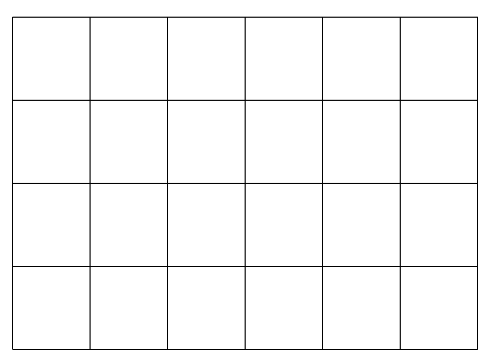
11. Find the distance between the circle $(x - 5)^2 + (y + 3)^2 = 1$ and the line $5x + 12y - 4 = 0$.

(A) $\frac{1}{13}$ **(B)** $\frac{2}{13}$ **(C)** $\frac{3}{13}$ **(D)** $\frac{3}{15}$ **(E)** $\frac{4}{15}$

12. How many rectangles *that are not squares* are in this grid? For example, the entire 4×6 grid is a rectangle but not a square; any $n \times n$ rectangle is a square and should not be counted.

(A) 160 **(B)** 170 **(C)** 185 **(D)** 200

(E) 210



13. If $a = \log 12$, $b = \log 21$, $c = \log 11$, and $d = \log 22$, then $\log\left(\frac{1}{7}\right)$ can be expressed in the form $P(a-b) + Q(c-d)$, where P and Q are integers. Find $7P + Q$.

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

14. Given that $\cos x + \cos y + \cos z = \sin x + \sin y + \sin z = 0$, find the numerical value of

$$\cos(x-y) + \cos(y-z) + \cos(z-x).$$

(A) $-\frac{3}{2}$ **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

15. Two brothers sold a flock of sheep that they owned. The number of dollars they received for each sheep was equal to the number of sheep in the herd. The buyer of the flock paid the brothers by giving the older brother \$10, then the younger brother \$10, then the older brother \$10, and so on, until all of the money was gone. The last brother to receive money was the younger brother, who received less than \$10, so the older brother gave him some money to make their amounts equal. How much money was given?

(A) \$1 **(B)** \$2 **(C)** \$3 **(D)** \$4 **(E)** \$5

16. There are infinitely many sequences whose first four terms are 2, 0, 0, 7. If f is the polynomial function of smallest degree such that $f(1) = 2$, $f(2) = f(3) = 0$, and $f(4) = 7$, find $f(5)$.

(A) 15 **(B)** 20 **(C)** 21 **(D)** 24 **(E)** 26

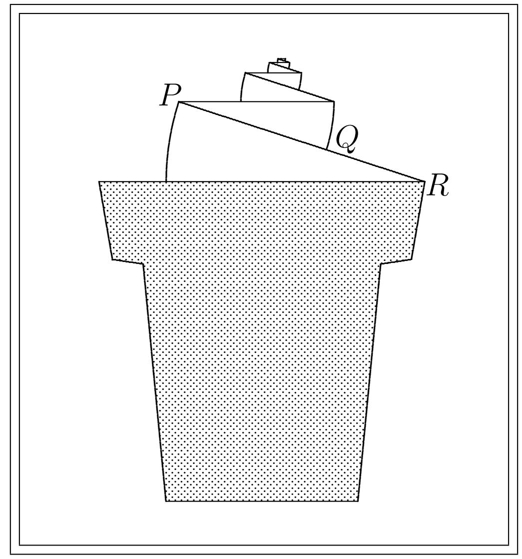
17. In right triangle MNP, Y is the midpoint of $\overline{MN}$. If hypotenuse $\overline{NP}$ has length 17, then $\overline{YP}$ cannot be shorter than p or greater than q. Find $p + q$.

(A) 24 **(B)** 25 **(C)** 25.5 **(D)** $17 + \frac{\sqrt{10}}{4}$ **(E)** $17 + \frac{17\sqrt{10}}{4}$

18. Emma has ten apples, and wants to give them to (up to) three friends. How many different ways can she do so? (Assume that the apples are all identical, but her friends are not; it is possible, for example, for her to give one friend all ten apples.)

(A) 36 **(B)** 63 **(C)** 66 **(D)** 1000 **(E)** 3^{10}

19. A soft-serve ice cream cone is shown in a photograph. Each three-dimensional swirl of ice cream (visible as a sector in the photograph) is similar to every other; in fact, any two neighboring swirls together are similar to any other two neighboring swirls together, and the swirls continue upward forever. If the volume of the bottom swirl is $\mathcal{S}$, the volume of ice cream in the cone is $\mathcal{C}$, and $QR = \frac{2}{5}PR$, find the total volume of ice cream.



(A) $\mathcal{C} + \frac{25}{21}\mathcal{S}$ **(B)** $\mathcal{C} + \frac{125}{98}\mathcal{S}$

(C) $\mathcal{C} + \frac{25}{16}\mathcal{S}$ **(D)** $\mathcal{C} + \frac{5}{3}\mathcal{S}$

(E) $\mathcal{C} + \frac{5}{2}\mathcal{S}$

20. For any number a, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a, and let $\{a\}$ denote the fractional part of a; in other words, $\{a\} = a - \lfloor a \rfloor$. For example, $\lfloor 5.3 \rfloor = 5$ and $\{5.3\} = 0.3$. Given that

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 4.2 \\ y + \lfloor z \rfloor + \{x\} &= 5.8 \\ z + \lfloor x \rfloor + \{y\} &= 7.4, \end{aligned}$$

find $2x + 3y + z$.

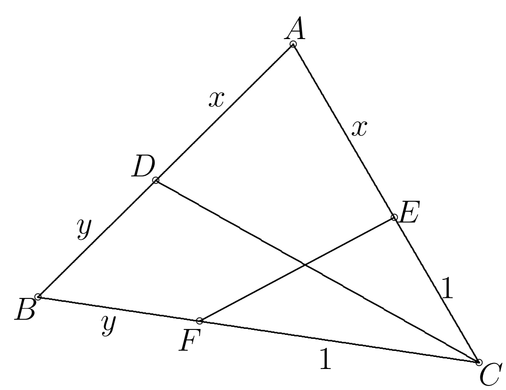
(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

21. In triangle ABC, D lies on $\overline{AB}$, E lies on $\overline{AC}$, and F lies on $\overline{BC}$. The segments shown have lengths in the following ratios:

$$AE : EC = x : 1$$

$$BF : FC = y : 1$$

$$AD : DB = x : y$$



What must be true about x and y in order that $\overline{EF}$ bisects $\overline{CD}$?

(A) $x = y$ **(B)** $x^2 = y^2$ **(C)** $xy = 1$

(D) $x + y = 2$ **(E)** $2xy = x + y$

22. Triangle ABC is inscribed in circle O of radius one. Angle A measures 30° and angle B measures 70°. M and N are points on circle O such that M is closer to C than to B and $\overline{AM}$ and $\overline{AN}$ trisect $\angle A$. The area of quadrilateral $BNMC$ is $P \sin 20^\circ - \sqrt{Q}$, where P and Q are positive rational numbers in simplest form. Find $16(P+Q)$.

(A) 21 **(B)** 23 **(C)** 25 **(D)** 27 **(E)** 29

23. A dimwitted painter is hired to paint a 200-foot line in the center of a straight road. His 200-step procedure is as follows:

   1. He fills his brush, paints one foot of the line, then walks one foot back to the paint can.
   2. He fills his brush, walks one foot to the end of the painted line, paints another foot, and walks two feet back to the paint can.
   3. He fills his brush, walks two feet to the end of the painted line, paints another foot, and walks three feet back to the paint can.

   (etc.)

The painter never moves the can!

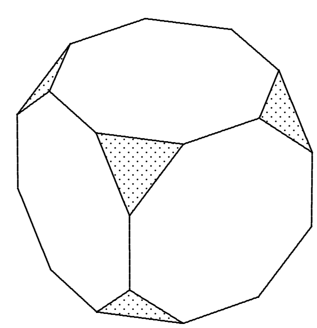
In the first hour, the painter is able to complete 100 of these steps (painting 100 feet of the line). In the second hour, because of the additional walking distance, he is only able to complete 50 more steps (painting 50 more feet of the line).

Assuming his rates of walking and painting are constant, that filling the brush takes no time, and that he does not rest or pause, how long will it take to complete all 200 steps?

(A) between 3 and 3.5 hours **(B)** between 3.5 and 4 hours **(C)** between 4 and 4.5 hours

(D) between 4.5 and 5 hours **(E)** more than 5 hours

24. The corners of a cube are sliced off, resulting in the solid shown to the right. If all edges of the solid have length 1, find the volume of the solid.

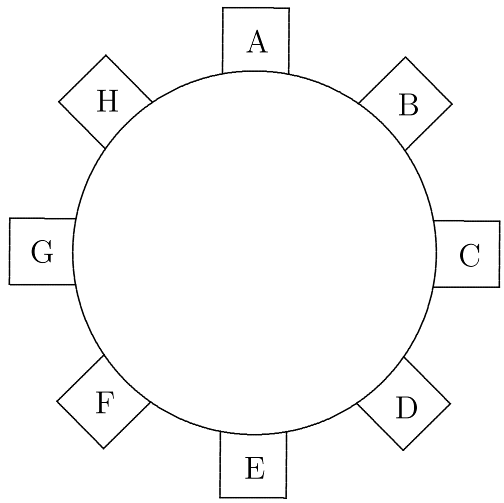


(A) $\frac{80}{3}$ **(B)** $27 - \frac{\sqrt{2}}{3}$ **(C)** $\frac{20}{3} + 5\sqrt{2}$

(D) $7 + \frac{14}{3}\sqrt{2}$ **(E)** $7 + 5\sqrt{2} - \frac{\sqrt{3}}{3}$

25. Chan, Daniel, and six other math students are sitting at a round table, each one thoroughly engrossed in a different math topic. The table has eight seats, labeled "A" through "H."

Baiyang
Chan
Christine
Daniel
Haisha
Michael
Shuyi
Tom



Algebra
Calculus
Combinatorics
Geometry
Graph Theory
Logic
Number Theory
Trigonometry

(a) Michael is directly opposite Haisha and to the immediate left of Shuyi, who is directly opposite the student who is working on combinatorics, who, in turn, is on the immediate right of the student who is studying number theory.

(b) The student solving a logic problem is in chair F. Tom is in chair G, directly opposite the student who is working on algebra (who is not Baiyang).

(c) The student studying graph theory is on the immediate right of the one working on a geometry problem, who is not sitting in chair E.

(d) Christine, who is not in chair D, is not the one who is studying trigonometry.

(e) The student in chair A is working on calculus.

(f) No student's name has the same initial as the chair in which he or she is sitting.

Determine which lettered seat is being used by Daniel.

(A) Chair A **(B)** Chair B **(C)** Chair C **(D)** Chair E **(E)** Chair H

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1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the ParSCORE Score Sheet with a #2 pencil. Check your blackened ovals for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded. The scoresheet has numbers up to 200; leave numbers 26 to 200 blank.
4. SCORING: You will receive 6 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer. Note that random guessing will almost certainly lower your score. If you can eliminate more than two answers, guessing might be advantageous.
5. You are permitted to use scratch paper, graph paper, rulers, compasses, protractors, erasers and most calculators. No problems *require* the use of a calculator.
6. Calculators with a QWERTY keyboard, such as the TI-92 or HP-95, or which contain a Computer Algebra System, such as the TI-89, HP40G, HP48GII, HP49G, or HP50G, are not allowed and must be given to your proctor for the entirety of the contest. Students with such a calculator near their desk, even if it is not used, will be disqualified.
7. Figures are not necessarily drawn to scale.
8. Before beginning the contest, your proctor will ask you to record specific information on the answer form. Specifically:
   - Your **I.D. NUMBER** is your school's six-digit CEEB code followed by a three digit student ID number, of the form AAAAAABBB. Leave the last column blank.
   - For your **NAME**, enter your complete last name, skip one space, then enter as much of your first name as you can. If your entire first name fits, skip a space and enter your middle initial, if it fits. If your first name and middle initial do not fit, you will hand-write them elsewhere.
   - For your **CODE**, in the **first column** enter

A	if you are in grade 8 or below
B	if you are in grade 9
C	if you are in grade 10
D	if you are in grade 11
E	if you are in grade 12
F	if you have graduated from high school (and are completing a postgraduate year)

   - Also for your **CODE**, in the **second column** enter Y if you were able to fit your complete name on the scoresheet; otherwise enter N. If you enter N, and score within the top 200 in the state, your handwritten name will be used. For example, the code "CY" specifies that you are a sophomore whose name fit completely; "EN" specifies that you are a senior whose name did not fit completely.
   - For **TEST FORM**, enter A.
   - Leave the **PHONE NUMBER** and **SUBJECT CODE** blank.
   - Turn your paper clockwise and **carefully hand-write your name and today's date.**

   Underneath each letter or number, blacken the corresponding oval.
9. When your proctor gives the signal, begin working on the problems. You will have **90 MINUTES** to complete the contest.