FORTY-THIRD ANNUAL OLYMPIAD

HIGH SCHOOL PRIZE COMPETITION

IN MATHEMATICS

2006 – 2007

FIRST LEVEL EXAMINATION

Tuesday, October 24, 2006

SOLUTIONS

Solution 1. Answer: $\frac{3}{7}$

Solution 2. Suppose Allison starts off with \$$a$ and Ben starts off with \$$b$, then $a+15\cdot5=b+15\cdot7$. Furthermore, $(b+30\cdot7)\cdot7=(a+30\cdot5)\cdot8$. We find $a = 60$ and $b=30$. Answer: 90

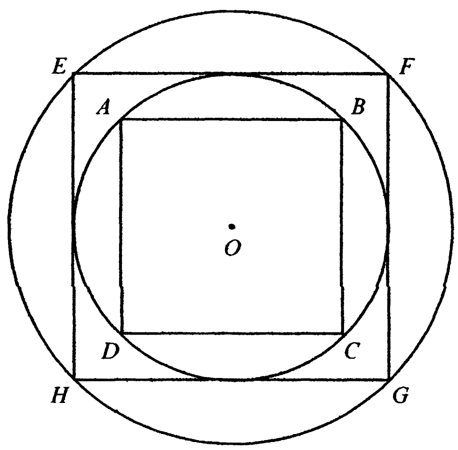
Solution 3. Let x equal the number of bowls Krishna orders. Then, we solve $\frac{5-0.1(x-1)}{x}=24$ to get x=15.

Solution 4. The answer is $16\cdot\cos^4 30^\circ=9$. Note: there is a non-trig solution to this problem.

Solution 5. Let x equal Jerry's speed. Solve $\frac{400}{x}-\frac{400}{x+1}=20$ to get $x = 4$.

Solution 6. Let x equal the number of juniors. Solve $\frac{15\cdot95+x\cdot75}{15+x}=82.5$ to get $x=25$.

Solution 7. Use the Pythagorean Theorem on ΔABC in the following picture.



Square $EFGH$ has side length $k\sqrt{2}$. Therefore, $EG = 2k$ and the answer is $2k\pi$.

Solution 8. The distance he travels does not change if she decides to travel along a different path (Why?). Any path is equivalent to Angela's walking along the two legs of an isosceles right triangle whose hypotenuse is the segment joining (0,0) and (10,0). Therefore, the minimum distance is $5\sqrt{2}+5\sqrt{2}=10\sqrt{2}$.

Solution 9. Since $1+(x+3)=x+4=y$ and $2+(x+2)=x+4=y+1$, clearly $x+4\geq16$. Thus, since $1+(x+3)$ carries, $1+1+(x+3)$ also carries, either once or twice. Therefore, either $1+x+(x+1)=10+16=26$ or $2+x+(x+1)=26$. Since the latter implies that x is not an integer, $2x+2=26$ and $x=12$.
From this, we know that $x+4=16$, y must equal 0 and $x+y=12$.

Solution 10. By the divisibility rules for three and for eleven, *BCDA* has the same remainder as *ABCD* when divided by 33. Thus, the answer is the remainder when $13 \cdot 3$ is divided by 33, which is 6.

Solution 11. Let a be the number that John subtracts. Then

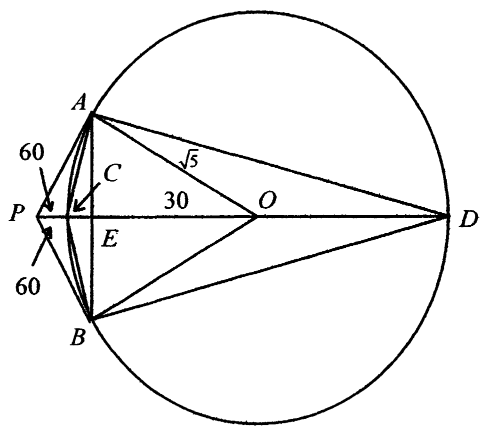
$$1+2+3+\cdots 200-2a=19904.$$

Therefore, $a=\dfrac{19904-\dfrac{200\cdot 201}{2}}{2}=98$.

Solution 12. $6(0.8\cdot 0.3+0.2\cdot 0.7)^2(0.2\cdot 0.3+0.8\cdot 0.7)^2=0.33304416$

Solution 13. Let $x=\log_3 y$. Since $\log_3 y=\dfrac{1}{\log_y 3}$, $\log_y 3=\dfrac{1}{x}$. Hence, we want to find solutions to $1=\dfrac{6}{x}-x$, which we solve to get $x=-3$ or $x=2$. Hence, $y=3^{-3}$ or 3^2. The answer is $9+\dfrac{1}{27}=\dfrac{244}{27}$.

Solution 14. Let E be the intersection of $\overline{AB}$ and $\overline{PD}$.



Hence, our answer is

$$\begin{aligned}\frac{1}{2}AB\cdot DE-\frac{1}{2}AB\cdot CE&=\frac{1}{2}AB(OD+OE-OC+OE)\\&=\frac{1}{2}AB(2OE)\\&=\frac{1}{2}\sqrt{5}\left(2\frac{\sqrt{15}}{2}\right)=\frac{5\sqrt{3}}{2}\end{aligned}$$

Solution 15. If $(x-1)^2+(y-2)^2=(x-1)(y-2)$, then
$(x-1)^3+(x-1)(y-2)^2=(x-1)^2(y-2)$ and
$(x-1)^2(y-2)+(y-2)^3=(x-1)(y-2)^2$. Summing these two equations, we get
$(x-1)^3+(y-2)^3=0$, which implies that $x=1$ and $y=2$. Therefore, the answer is 9.

Solution 16. If $q|p^q$, then $q|p$ because both numbers are prime. This is possible only when $q=p$. Since there are 4 one-digit primes, the answer is $\frac{4}{16}=\frac{1}{4}$.

Solution 17. Notice that $B=(6\cos(45),6\sin(45))$; therefore
$C=(6\cos(45\pm 60),6\sin(45\pm 60))$. Hence, the answer is
$6\sin(105)+6\sin(-15)=3\sqrt{2}$.

Solution 18. Notice that ΔAEF has twice the base and half the height of ΔFED (Why?). Answer: $\frac{K}{6}$

Solution 19. $x^2y=y^2x$ implies $x^2y-y^2x=0$ or $xy(x-y)=0$, which in turn implies that $x=0$, $y=0$, or $x=y$. Plugging these values into $(x+y)^2=4$, we get six points on a circle of radius 2, centered at the origin, namely
$(2,0),(\sqrt{2},\sqrt{2}),(0,2),(-2,0),(-\sqrt{2},-\sqrt{2})$, and $(0,-2)$. We can interpret the area of the convex polygon formed from the solution set of this system as two trapezoids. Hence, the answer is $2\cdot\frac{4+2\sqrt{2}}{2}\cdot\sqrt{2}=4\sqrt{2}+4$.

Solution 20. Notice that

$$\begin{aligned}\left(3^{\frac{n+3}{3}}-3^{\frac{n}{3}}\right)^3&=\left(3^{\frac{n}{3}+1}-3^{\frac{n}{3}}\right)^3\\&=3^{n+3}-3\cdot 3^{\frac{3n}{3}+2}\cdot 3^{\frac{n}{3}}+3\cdot 3^{\frac{n}{3}+1}\cdot 3^{\frac{2n}{3+1}}-3^n\\&=3^{n+3}-3^{n+3}+3^{n+2}-3^n\\&=3^{n+2}-3^n\end{aligned}$$

Hence, our answer is $\sum_{n=1}^{2003}3^{n+2}-3^n=3^{2005}+3^{2004}-3^2-3=12\left(3^{2003}-1\right)$.

Solution 21. Notice that

$$\begin{aligned}64 &= x^4+8x^2+24+8y^2+y^4=64\\ &= x^2+4(xy)x^2+6(xy)^2+4(xy)y^2+y^4\\ &= (x+y)^4\end{aligned}$$

Hence, for real *x* and *y*, $x+y=\pm 2\sqrt{2}$; therefore,

$x^2+2xy+y^2-2xy=\left(\pm 2\sqrt{2}\right)^2-4=4$

Solution 22. Set real and imaginary parts of each side of each equation equal and use the identity $\sin^2\theta+\cos^2\theta=1$ to derive the answer $2\left(x^2+y^2\right)$.

Solution 23. We calculate $f(2)=a, f(3)=1/b, f(4)=a/b, f(5)=a/b^2$, and $f(6)=a^2/b^3$. Notice that the powers of a and b become Fibonacci numbers. The answer is a^{144}/b^{233}.

Solution 24. The probability that Jack and Jill will not see each other on the n^{th} day from now is

$$1-\frac{3}{3n+5}=\frac{3n+2}{3n+5}=\frac{3n+2}{3(n+1)+2}.$$

Hence, by using telescoping products, we find that our answer is

$$\prod_{i=1}^{31}\frac{3n+2}{3(n+1)+2}=\frac{3(1)+2}{3(32)+2}=\frac{5}{98}$$

Solution 25. $f(n)=2^n$ because each of the *n* court jesters may be either included or not included. Therefore,

$$\begin{aligned}\log_2 f(1)+\log_2 f(2)+\cdots+\log_2 f(100) &= \log_2 2^1+\log_2 2^2+\cdots+\log_2 2^{100}\\ &= 1+2+3+\cdots+100=(100\cdot 101)/2=5050\end{aligned}$$

Solution 26. Note that

$$\left(b^*\right)^2-b^2=\left(b-\lfloor b\rfloor\right)^2-b^2=b^2-2b\lfloor b\rfloor+\lfloor b\rfloor^2-b^2=\lfloor b\rfloor^2-2b\lfloor b\rfloor.$$

Since $\lfloor b\rfloor^2$ is an integer by definition, we need only ensure that $2b\lfloor b\rfloor$ is an integer. We will work with positive *b* first. Clearly, *b* must be rational and $\lfloor b\rfloor\in\{1,2,3,4\}$. If $\lfloor b\rfloor=1$, then we have to consider only those *b* with denominator $2\cdot 1=2$. Similarly, if $\lfloor b\rfloor=2$, we look at only those *b* with denominator $2\cdot 2=4$. We also observe that $\lfloor b\rfloor\le b<\lfloor b\rfloor+1$. We can now list the possible *b* for $b\ge 1$:

$$\frac{2}{2},\frac{3}{2},\frac{8}{4},\frac{9}{4},\ldots,\frac{11}{4},\frac{18}{6},\frac{19}{6},\ldots,\frac{23}{6},\frac{32}{8},\frac{33}{8},\ldots,\frac{39}{8}.$$

We find a total of 20 positive *b*. For negative *b*, we derive these possibilities for b:

$$-\frac{2}{2},-\frac{3}{2},-\frac{5}{4},-\frac{6}{4},\dots,-\frac{8}{4},-\frac{13}{6},-\frac{14}{6},\dots,-\frac{18}{6},-\frac{25}{8},-\frac{26}{8},\dots,-\frac{32}{8},-\frac{41}{10},-\frac{42}{10},\dots,-\frac{49}{8}$$

Hence, the answer is 20 + 29 = 49.

Solution 27. Let the angle bisector of $\angle C$ intersect $\overline{AB}$ at D. Notice that $\Delta ABC \sim \Delta CBD$. Thus, $AD = DC = \frac{ab}{c}, \frac{DB}{a} = \frac{a}{c}$. Combining these results, we derive

$$a^2 + ab = c^2$$

Therefore, $c = 77$.

Solution 28. There are two cases: for $-2 \le x \le 2$,, either $|f(x)| = f(x)$; or $|f(x)| = -f(x)$. The first leads to a contradiction; therefore, $|f(x)| = -f(x)$. We know $2f(x) = f(x) + g(x) - \left(|f(x)| + g(x)\right) = 2x^2 + 2x + 1 - (2x+9)$, which implies $f(x) = x^2 - 4$ and $g(x) = 2x^2 + 2x + 1 - \left(x^2 - 4\right) = x^2 + 2x + 5$. Hence, $f(x) + 2g(x) = 3x^2 + 4x + 6$, whose minimum occurs at $x = -\frac{4}{2\cdot 3} = -\frac{2}{3}$. Since $\left(f\left(-\frac{2}{3}\right) + 2g\left(-\frac{2}{3}\right)\right) = \frac{14}{3}$, the minimum value of $|f(x) + 2g(x)|$ is also $\frac{14}{3}$.

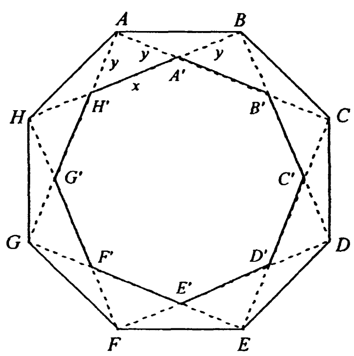
Solution 29. Isolate the 12 and then divide by xyz to get

$$12/xyz = 1/x + 1/y + 1/z$$

$$\le 3\sqrt[3]{1/xyz}, \text{ by the AM-GM inequality,}$$

which implies $4 \le (xyz)^{\frac{2}{3}}$ or $8 \le xyz$.. Answer: 8.

Solution 30. $y = AA'$ and $x = A'H'$.



By the Law of Cosines, $1^2 = y^2 + y^2 - 2y^2 \cos 135$, which implies that $y^2 = \frac{1}{2+\sqrt{2}}$. By the Pythagorean Theorem, $\Delta AA'H'$, $2y^2 = x^2 = \frac{2}{2+\sqrt{2}}$. If K is the area of a regular octagon of side 1, then our answer is

$K + Kx^2 + Kx^4 + \cdots = \frac{K}{1-x^2}$. But, $K = 2 + 2\sqrt{2}$;

hence, our answer is $\dfrac{2+2\sqrt{2}}{1-\dfrac{2}{2+\sqrt{2}}} = 4\sqrt{2} + 6$, which is closest to 12.