FORTY-THIRD ANNUAL OLYMPIAD

HIGH SCHOOL PRIZE COMPETITION

IN MATHEMATICS

2006 – 2007

Conducted by

The Massachusetts Association
of
Mathematics Leagues
(MAML)

Sponsored by

The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

Tuesday, October 24, 2006

1. Solve for x.

$$\frac{1}{2-\frac{1}{1+\frac{1}{1-\frac{1}{5}}}}=\frac{3}{2}x$$

(A) $\frac{1}{21}$ (B) $\frac{3}{7}$ (C) $\frac{5}{9}$ (D) $\frac{9}{14}$ (E) $\frac{5}{6}$

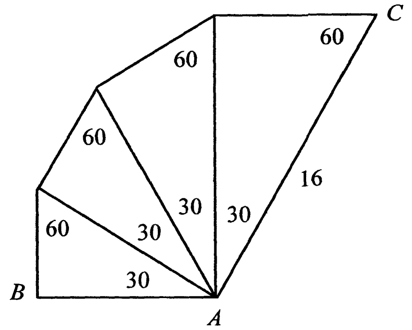
2. Allison and Ben are saving for college. Allison starts with \$a in her piggy bank and Ben starts with \$b in his. Each week, Allison deposits \$5 into her piggy bank and Ben deposits \$7 into his. In 15 weeks, they have the same amount. In 30 weeks, the ratio of Ben's savings to Allison's is 8:7. Determine $a+b$.

(A) 30 (B) 40 (C) 60 (D) 90 (E) 120

3. At a certain restaurant, the first bowl of tripe soup costs \$5; for each additional bowl of soup, the restaurant will pay the customer \$.10. How many bowls of tripe soup must Krishna eat if he wants to pay an average of \$.24 per bowl?

(A) 4 (B) 5 (C) 14 (D) 15 (E) 24

4. 30-60-90 triangles are constructed as in the following picture. If $AC = 16$, what is AB?



(A) 1 (B) 2 (C) $3\sqrt{3}$ (D) 9 (E) $6\sqrt{3}$

5. Ben and Jerry are running in a 400 meter race. Both racers run at a constant speed, and Ben runs 1 meter per second faster than Jerry. If Ben's time for the 400 meters is 20 seconds less than Jerry's time, which of the following is Jerry's speed in meters per second?

(A) 4 (B) 5 (C) 16 (D) 22 (E) 34

6. There are juniors and seniors in a math class. On the last test, the class average was 82.5%. The 15 seniors in the class had an average score of 95%. If the average for the juniors was 75%, how many juniors are there in the class?

(A) 9 (B) 15 (C) 20 (D) 25 (E) 40

7. Let *ABCD* be a square with side *k* inscribed in circle centered at *O*. Circle *O* is inscribed in square *EFGH*, and *EFGH* is inscribed in another circle which is also centered at *O*. Find the circumference of the larger circle.

(A) $k\sqrt{2}$ (B) πk (C) $2\pi k$ (D) $2\pi k\sqrt{2}$ (E) $4\pi k$

8. Angela wants to walk from the point (0,0) to the point (10,0). She may walk only along line segments whose slope is 1 or -1 . What is the minimum distance she can travel as she walks?

(A) $5\sqrt{2}$ (B) 10 (C) $10\sqrt{2}$ (D) 20 (E) $20\sqrt{2}$

9. All numbers in the following sum are integers in base 16. Find $x+y$ in base 10.

$$\begin{array}{rcccc} & & x & 2 & 1 \\ + & & (x+1) & (x+2) & (x+3) \\ \hline & 1 & 10 & (y+1) & y \end{array}$$

(A) 12 (B) 18 (C) 24 (D) 36 (E) 40

10. If *ABCD* is a four-digit positive integer in base 10 that is divisible by 33, what is the remainder when 13(*BCDA* + 3) is divided by 33?

(A) 3 (B) 6 (C) 7 (D) 13 (E) 12

11. John is adding up the first 200 positive integers with a calculator. He accidentally hits the minus key instead of the plus key for one of the numbers and gets a sum of 19904. Which number did he subtract?

(A) 96 (B) 98 (C) 99 (D) 196 (E) 198

12. For each of the next four days, there is an 80% chance that the weatherman will say that it will snow on that day. Each day, there is a 70% chance that the weatherman will be wrong. What is the probability (to the nearest thousandth) that it will snow on exactly two of the next four days?

(A) 0.056 (B) 0.058 (C) 0.154 (D) 0.333 (E) 0.346

13. Find the sum of all *y* such that $1 = 6\log_y 3 - \log_3 y$.

(A) -1 (B) 1 (C) 5 (D) $244/27$ (E) $244/9$

14. Given a circle of radius $\sqrt{5}$ and point *P* outside of the circle such that $\angle APB = 120^\circ$ where *A* and *B* are the points of tangency from *P* to the circle. The angle bisector of $\angle APB$ is $\overline{PCD}$, where *C* and *D* are the circle. Find $\left| Area(\Delta ABC) - Area(\Delta ABD) \right|$.

(A) $\frac{5\sqrt{3}+10}{4}$ (B) $\frac{5\sqrt{3}}{2}$ (C) $\frac{10\sqrt{3}+10}{4}$ (D) $5\sqrt{3}$ (E) $\frac{15\sqrt{3}+10}{4}$

15. If $(x-1)^2+(y-2)^2=(x-1)(y-2)$, and all variables are integers, determine x^3+y^3.

(A) 0 (B) $\frac{\sqrt{5}+1}{2}$ (C) 3 (D) 5 (E) 9

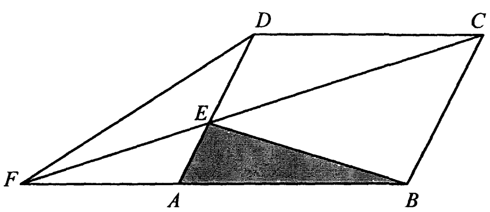
16. If p and q are one-digit primes, what is the probability that p^q is divisible by q?

(A) $\frac{4}{25}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{5}{16}$ (E) $\frac{1}{2}$

17. Given equilateral triangle ABC, where A=(0,0) and $B=\left(3\sqrt{2},3\sqrt{2}\right)$, there are two possible locations in the xy-plane for point C. Find the sum of all possible y-coordinates of C.

(A) $\frac{3\sqrt{2}-3\sqrt{6}}{2}$ (B) $3\sqrt{2}$ (C) $\frac{3\sqrt{2}+3\sqrt{6}}{2}$ (D) $3\sqrt{6}$ (E) $6\sqrt{2}$

18. Given parallelogram $ABCD$ of area K, let point E lie on $\overline{AD}$ such that $\text{Area}(\Delta AEB) = \frac{K}{6}$. Let segment $\overline{BA}$ extended intersect segment $\overline{CE}$ extended at F. Find $\text{Area}(\Delta FED)$.



(A) $\frac{K}{2}$ (B) $\frac{K}{3}$ (C) $\frac{K}{4}$ (D) $\frac{K}{5}$ (E) $\frac{K}{6}$

19. Consider the following system:

$$x^2y=y^2x$$
$$x^2+y^2=4.$$

The solutions (x,y) of this system form the vertices of a convex polygon. Find the area of this polygon.

(A) $2\sqrt{2}+2$ (B) $2\sqrt{2}+4$ (C) $4\sqrt{2}+4$ (D) $6\sqrt{3}$ (E) $8\sqrt{2}+8$

20. Find the value of $\left(3^{4/3}-3^{1/3}\right)^3+\left(3^{5/3}-3^{2/3}\right)^3+\left(3^{6/3}-3^{3/3}\right)^3+\cdots+\left(3^{2006/3}-3^{2003/3}\right)^3$

(A) $12\left(3^{2002}-1\right)$ (B) $8\left(3^{2003}+1\right)$ (C) $12\left(3^{2003}-1\right)$
(D) $3\left(3^{2004}-1\right)$ (E) $3\left(3^{2005}-1\right)$

21. Solve the following system and find x^2+y^2, where x and y are real:

$$x^4+8x^2+24+8y^2+y^4=64$$
$$xy=2$$

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4 (E) 8

22. Consider the system of equations

$$a\sin\theta+ib\cos\theta=x+iy$$
$$b\sin\theta+ia\cos\theta=y+ix$$

where $i=\sqrt{-1}$. If a, b, x, y, θ are real, what is a^2+b^2 in terms of x and y?

(A) $x+y$ (B) x^2+y^2 (C) $\sqrt{2}(x+y)$ (D) $\sqrt{2}\left(x^2+y^2\right)$ (E) $2x^2+2y^2$

23. For real numbers n, let f be a function with the property that $f(n+1)=f(n)f(n-1)$. If $f(0)=a^2b$ and $f(1)=\frac{1}{ab}$, where a and b are nonzero real numbers, find $f(15)$.

(A) $\frac{a^{89}}{b^{144}}$ (B) $\frac{1}{a^{89}b^{89}}$ (C) $\frac{a^{144}}{b^{233}}$ (D) $\frac{1}{a^{144}b^{144}}$ (E) $\frac{a^{233}}{b^{377}}$

24. Jack and Jill got into a fight. The probability that they will see each other on the n^{th} day from now is $\frac{3}{3n+5}$. What is the probability that they will not see each other for the next 31 days?

(A) $\frac{2}{95}$ (B) $\frac{3}{98}$ (C) $\frac{5}{98}$ (D) $\frac{5}{95}$ (E) $\frac{95}{98}$

25. King Kamehameha wants to slay a dragon. He can go on the mission alone, or he can take some number of court jesters with him. If he has n distinct court jesters, let $f(n)$ equal the number of ways in which Kamehameha can form an entourage from these court jesters, choosing none or one or more. Calculate
$\log_2 f(1)+\log_2 f(2)+\log_2 f(3)+\cdots+\log_2 f(100)$

(A) 4950 (B) 5050 (C) $2^{100}-1$ (D) $2^{101}-1$ (E) 2^{101}

26. Let $x^* = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ equals the greatest integer y such that $y \le x$. How many numbers b are there such that $\left(b^*\right)^2 - b^2$ is an integer and $1 \le |b| < 5$?

(A) 20 (B) 40 (C) 49 (D) 50 (E) 58

27. Suppose ΔABC has the property that $\angle C = 2\angle A$. If side a = 49 and side b = 72, find side c.

(A) $49\sqrt{2}$ (B) 77 (C) 87 (D) 98 (E) 100

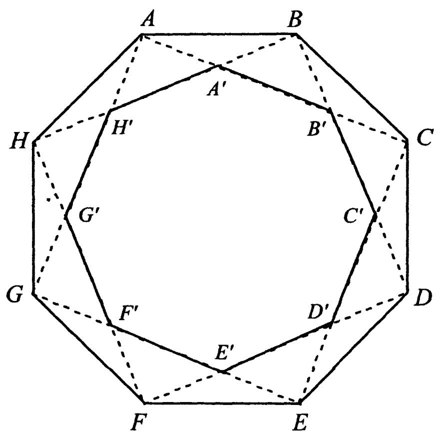
28. Let f and g be polynomials defined for all real numbers. For $-2 \le x \le 2$, $|f(x)| + g(x) = 2x + 9$ and for $x > 2$, $f(x) + g(x) = 2x^2 + 2x + 1$. Find the minimum value of $|f(x) + 2g(x)|$.

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{14}{3}$ (E) 5

29. If x, y, and z are positive real numbers and $x = \dfrac{12 - yz}{y + z}$, find the maximum value of xyz.

(A) 2 (B) 4 (C) 8 (D) $24\sqrt{3}$ (E) 64

30. Let $ABCDEFGH$ be a regular octagon with sides of length 1. Draw $\overline{AC}$, $\overline{BD}$, $\overline{CE}$, etc. to form a new octagon $A'B'C'D'E'F'G'H$.



We perform the same process on the new smaller octagon to form an even smaller octagon. If we continue this process indefinitely, which of the following is closest to the sum of the areas of all interior octagons formed, including $ABCDEFGH$?

(A) 5 (B) 12 (C) 16 (D) 21 (E) 24