Problem 1: How Big Is That Polygon?

a $\boxed{1, 1}$ When $n = 4$ the polygon is a unit square, so $A_1A_4 = 1$ and the area is 1.

b $\boxed{\sqrt{3}, 1.25\sqrt{3}}$ When $n = 5$, $\angle A_2 + \angle A_3 + \angle A_4 = 360°$, so $\angle A_2 = \angle A_3 = \angle A_4 = 120°$. Let P be the midpoint of A_2A_4, so triangles A_2A_3P and A_4A_3P are 30°60°90° triangles and $A_1A_5 = 2A_2P = 2\left(\sqrt{3}/2\right) = \sqrt{3}$. Now the area

$$[A_1A_2A_3A_4A_5] = [A_1A_2A_4A_5] + [A_2A_3A_4] = \left(1 \cdot \sqrt{3}\right) + \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \sqrt{3}\right) = 5\frac{\sqrt{3}}{4}$$

c $\boxed{1+\sqrt{2},\ \ 2.5+2.5\sqrt{2}}$ When $n = 6$, $\angle A_2 + \angle A_3 + \angle A_4 + \angle A_5 = 540°$, so $\angle A_2 = \angle A_3 = \angle A_4 = \angle A_5 = 135°$. Let P and Q be the feet of the perpendiculars on $\overline{A_2A_5}$ from A_3 and A_4, respectively. Then triangles A_2A_3P and A_5A_4Q are isosceles right triangles triangles and

$$A_1A_6 = A_2P + PQ + QA_5 = \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} = 1 + \sqrt{2}.$$

The area

$$[A_1A_2A_3A_4A_5A_6] = [A_1A_2A_5A_6] + 2[A_2A_3P] + [A_3A_4QP] = \left(1 \cdot \left(1 + \sqrt{2}\right)\right) + 2\left(\frac{1}{4}\right) + \frac{\sqrt{2}}{2} = \frac{3}{2} + \frac{3\sqrt{2}}{2}$$

Problem 2: Graph the Greatest.

a (**i**) This graph is the union of the following six line segments:

$$\{(x,9): -3 < x < -2\} \quad \{(x,4): -2 \le x < -1\} \quad \{(x,1): -1 \le x < 0\}$$
$$\{(x,0): 0 \le x < 1\} \quad \{(x,1): 1 \le x < 2\} \quad \{(x,4): 2 \le x < 3\}$$

(**ii**) This graph is the union of the following seventeen line segments:

$$\left\{(x,n): -\sqrt{n+1} < x \le -\sqrt{n}\right\} \quad \text{for } n = 8,7,6,\ldots,1$$
$$\{(x,0): -1 < x < 1\}$$
$$\left\{(x,n): \sqrt{n} \le x < \sqrt{n+1}\right\} \quad \text{for } n = 1,2,3,\ldots,8$$

(**iii**) This graph is the union of the following seventeen line segments:

$$\left\{\left(\sqrt{|n|}, y\right) : n \le y < n+1\right\} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots, \pm 8.$$

(**iv**) This graph is the union of the following six squares:

$$\{(x,y): -3 < x < -2,\ 9 \le y < 10\}$$
$$\{(x,y): -2 \le x < -1,\ 4 \le y < 5\} \quad \{(x,y): 2 \le x < 3,\ 4 \le y < 5\}$$
$$\{(x,y): -1 \le x < 0,\ 1 \le y < 2\} \quad \{(x,y): 1 \le x < 2,\ 1 \le y < 2\}$$
$$\{(x,y): 0 \le x < 1,\ 0 \le y < 1\}$$

b $\boxed{12}$ These are the twelve pairs of integers that sum to 25:

$$(m,n) \in \{(0,\pm 5), (\pm 5, 0), (\pm 3, \pm 4), (\pm 4, \pm 3)\}$$

and for each pair (m,n) there is a unit square

$$\{(x,y): m \le x < m+1,\ n \le y < n+1\}$$

in the graph, so the area of the graph is 12.

c $\boxed{1,2,3,4,6,7,8,9,11,12,14,16,18,19}$ Let u and v denote integers. Call $(m,n) = (0, \pm k)$ and $(m,n) = (\pm k, 0)$ *trivial solutions* to $u^2 + v^2 = k^2$. The area of $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = k^2$ will be four if and only if the only solutions to $u^2 + v^2 = k^2$ are the trivial solutions.

- The Pythagorean Triples with hypotenuse $k <= 20$ are

$$\begin{aligned}(u,v,k) = {} & (3,4,5),\ (6,8,10),\ (9,12,15),\ (12,16,20), \\ & (5,12,13), \\ & (8,15,17).\end{aligned}$$

Thus the area will be four if and only if $k \ne 5, 10, 13, 15, 17, 20$.

Problem 3: My Calculator Broke $\frown$

a Since the product of a one-digit with a two-digit number is at most a three-digit number and no number under 20 is the product of a prime and a two-digit number, we concern ourselves only with numbers of the form $10d$ and $1d0$ with $d \in \{1, 2, 3, \ldots, 9\}$. We need not consider 101, 103, 107, and 109 which are all primes, so none of them has two-digit factor.

d	Display = 10d	Display = 1d0
1		$110 = 2{\cdot}5{\cdot}11 = \begin{cases} 2{\cdot}55 \\ 5{\cdot}22 \end{cases}$
2	$102 = 2{\cdot}3{\cdot}17 = \begin{cases} 2{\cdot}51 \\ 3{\cdot}34 \end{cases}$	$120 = 2^3{\cdot}3{\cdot}5 = \begin{cases} 2{\cdot}60 \\ 3{\cdot}40 \\ 5{\cdot}24 \end{cases}$
3		$130 = 2{\cdot}5{\cdot}13 = \begin{cases} 2{\cdot}65 \\ 5{\cdot}26 \end{cases}$
4	$104 = 2^3{\cdot}13 = 2{\cdot}52$	$140 = 2^2{\cdot}5{\cdot}7 = \begin{cases} 2{\cdot}70 \\ 5{\cdot}28 \\ 7{\cdot}40 \end{cases}$
5	$105 = 3{\cdot}5{\cdot}7 = \begin{cases} 3{\cdot}35 \\ 5{\cdot}21 \\ 7{\cdot}15 \end{cases}$	$150 = 2{\cdot}3{\cdot}5^2 = \begin{cases} 2{\cdot}75 \\ 3{\cdot}50 \\ 5{\cdot}30 \end{cases}$
6	$106 = 2{\cdot}53$	$160 = 2^5{\cdot}5 = \begin{cases} 2{\cdot}80 \\ 5{\cdot}32 \end{cases}$
7		$170 = 2{\cdot}5{\cdot}17 = \begin{cases} 2{\cdot}85 \\ 5{\cdot}34 \end{cases}$
8	$108 = 2^2{\cdot}3^3 = \begin{cases} 2{\cdot}54 \\ 3{\cdot}36 \end{cases}$	$180 = 2^2{\cdot}3^2{\cdot}5 = \begin{cases} 2{\cdot}90 \\ 3{\cdot}60 \\ 5{\cdot}36 \end{cases}$
9		$190 = 2{\cdot}5{\cdot}19 = \begin{cases} 2{\cdot}95 \\ 5{\cdot}38 \end{cases}$

b $\boxed{b = 1 \text{ or } a = c}$

$$ab + c = a + bc \iff ab - bc = a - c \iff b(a - c) = a - c \iff b = 1 \text{ or } a = c.$$

Problem 4: Equal Rights For All Digits!

a *Cast out nines:* The sum of the digits of 71 is 8, so remainder when 71^2 is divided by 9 is the same as the remainders when 64 and when 10 are divided by 9, and that remainder is 1. Since $71^{43} = \left(71^2\right)^{21} 71$, when nines are cast out of 71^{43} the remainder is $1^{21}(7+1) = 8$.

Assume that no digit appeared more than 8 times in 71^{43}. Then each of the 10 digits would appear 8 times, and the sum of the digits of 71^{43} would be $8(0+1+2+\cdots+9) = 8(45)$ which leaves a result of 0 when nines are cast out, a contradiction. Therefore, some digit appears more than 8 times.

b $0+1+2+\cdots+9 = 45$. If each digit appears k times in m^n, then the sum of the digits is $45k$, which is divisible by 9. Consequently m^n is divisible by 9. Thus,

either m is divisible by 9,

or m is divisible by 3 and $n \geq 2$.

Note that these are necessary, but not sufficient conditions that m^n contain each digit an equal number of times.

Problem 5: Beautify With Balloons!

a $\boxed{120}$ In $\triangle ABC$, A, B, and C are all different colors, so there are 6 choices for A, 5 for B and 4 for C for a total of $6 \cdot 5 \cdot 4 = 120$ patterns.

b $\boxed{630}$ In the rectangle $ABCD$, either A and C are the same color or they are different colors.

⇒ If A and C are the same color, then there are 6 choices of colors for the one color used for both A and C and 5 choices remain for each of B and D, for a total of $6 \cdot 5^2 = 150$ choices.

⇒ If A and C are different colors, then there are 6 choices for A, 5 for B, and 4 for each of B and D, for a total of $6 \cdot 5 \cdot 4^2 = 480$ choices.

Thus there are a total of $150 + 480 = 630$ patterns.

c $\boxed{3120}$ In the pentagon $ABCDE$, either A and C are the same color or they are different colors. In the cases when they are different colors, A and D may be the same color or different colors. Thus, there are three cases:

⇒ If A and C are the same color, then there are 6 choices of colors for the identically-colored A and C, 5 choices B, and $5 \cdot 4$ choices for D and E for a total of $6 \cdot 5^2 \cdot 4 = 600$ choices.

⇒ If A and C are different colors and D is the same color as A, then there are 6 choices for the identically-colored A and D and 5 choices for C. Since A and D are the same color, there are 5 choices for E, and since A and C are different colors there are 4 choices for B for a total of $6 \cdot 5^2 \cdot 4 = 600$.

⇒ If A and C are different colors and D is still another color, then there are 6 choices for A, 5 choices for C and 4 for D. Since A and D are different colors, there are 4 choices for E, and since A and C are different colors there are 4 choices for B for a total of $6 \cdot 5 \cdot 4^3 = 1920$.

Thus there are a total of $600 + 600 + 1920 = 3120$ patterns.

d $\boxed{n(n-1)(n-2)^2(n-3) = n^5 - 8n^4 + 23n^3 - 28n^2 + 12n}$ Choose the color for the junctions in the order c, d, a, e, b. Choose one of the n colors for c, and then there are $n-1$ choices for d. Since both a and e are joined to both c and d but not to each other, there are $n-2$ choices for both a and e. Finally, since b is joined to a, c, and d, there are $n-3$ choices for b. Thus $f(n) = n(n-1)(n-2)^2(n-3)$.

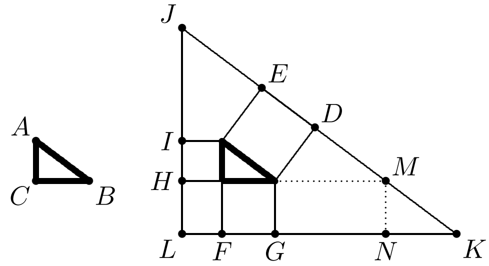
Problem 6: What Pythagoras Wondered About.

Without loss of generality, the vertices in both $\triangle ABC$ and $\triangle JKL$ are named in clockwise order and $\overline{JK} \parallel \overline{AB}$, $\overline{JL} \parallel \overline{AC}$, and $\overline{KL} \parallel \overline{BC}$ so that $\triangle ABC \sim \triangle JKL$.

a $\boxed{3+6\sqrt{3}}$ Since $\triangle JAI$ is a 30°-60°-90° triangle with $AI = 1$ and $IJ = \sqrt{3}$. The total perimeter of $\triangle JKL$ equals $3IH + 6IJ = 3 + 6\sqrt{3}$.

b $\boxed{10+5\sqrt{2}}$ Extend $\overline{BC}$ to intersect segment $\overline{DK}$ at M. Then $DM = BD = \sqrt{2}$ and $MK = BG\sqrt{2} = \sqrt{2}$. Consequently, $JK = ED + 2DM + 2MK = 5\sqrt{2}$. Since $\triangle ABC \sim \triangle JKL$ and $AB = \sqrt{2}$, the perimeter of $\triangle JKL$ is 5 times the perimeter of $\triangle ABC$, or $5\left(1+1+\sqrt{2}\right)$.

c $\boxed{62}$ Let $\overline{BC}$ intersect segment $\overline{DK}$ at M. Let N be the foot of the perpendicular from M to segment $\overline{GK}$.

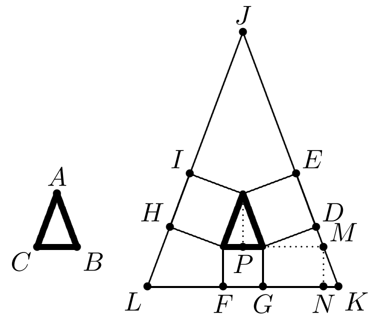


Then $\triangle ABC \sim \triangle BMD \sim \triangle MKN$ Consequently, $BM = \dfrac{AB}{AC}BD = \dfrac{25}{3}$ and $NK = \dfrac{CB}{AC}MN = \dfrac{16}{3}$. Thus

$$LK = LF + FG + GN + NK = 3 + 4 + \frac{25}{3} + \frac{16}{3} = \frac{62}{3}.$$

Since the perimeters are in the ratio $CB : LK$, the perimeter of $\triangle JKL$ is $\frac{62/3}{4} \cdot (3+4+5) = 62$.

d $\boxed{(2b+a) + \frac{(2b^2+a^2)(2b+a)}{a\sqrt{b^2-\left(\frac{a}{2}\right)^2}}}$ Construct auxiliary lines similar to those in the previous part and, in addition, let P be the midpoint of $\overline{BC}$.



Since $\triangle ABP \sim \triangle BMD \sim \triangle MKN$, we have $BM = \frac{AB}{AP}BD = \frac{b}{h}b = \frac{b^2}{h}$ and $NK = \frac{PB}{AP}MN = \frac{a/2}{h}a = \frac{a^2}{2h}$ where $h = AP = \sqrt{b^2 - \left(\frac{a}{2}\right)^2}$. Since $LF = GK$, it follows that

$$LK = FG + 2GK = FG + 2(GN + NK) = a + 2\left(\frac{b^2}{h} + \frac{a^2}{2h}\right) = \frac{ah + 2b^2 + a^2}{h}.$$

Since ratio of the perimeter of $\triangle JKL$ to the perimeter of $\triangle ABC$ is $KL : BC$, our answer is

$$\frac{\frac{ah+2b^2+a^2}{h}}{a}(2b+a) = \frac{(ah+2b^2+a^2)(2b+a)}{ah} = \left(1 + \frac{2b^2+a^2}{a\sqrt{b^2-\left(\frac{a}{2}\right)^2}}\right)(2b+a).$$

Note: Both parts **a** and **b** above are special cases of isosceles triangles, so a partial check of our work is to verify our answers to those previous parts using this formula:

a $a = b = 1$: $(2b+a) + \dfrac{(2b^2+a^2)(2b+a)}{a\sqrt{b^2 - \left(\frac{a}{2}\right)^2}} = 3 + \dfrac{(2+1)(2+1)}{1\sqrt{1-\frac{1}{4}}} = 3 + \left(\dfrac{9}{\sqrt{3}/2}\right) = 3 + 6\sqrt{3}.$

b $a = \sqrt{2},\ b = 1 : (2b+a) + \dfrac{(2b^2+a^2)(2b+a)}{a\sqrt{b^2 - \left(\frac{a}{2}\right)^2}} = \left(2+\sqrt{2}\right) + \dfrac{(2+2)\left(2+\sqrt{2}\right)}{\sqrt{2}\sqrt{1-\frac{1}{2}}}$

$$= 2 + \sqrt{2} + 8 + 4\sqrt{2} = 10 + 5\sqrt{2}.$$

Problem 7: Eight Is The Number.

a $n^2 - 1 = (n-1)(n+1)$ and $n-1$ and $n+1$ are consecutive even integers. In every pair of consecutive even integers, exactly one has a factor of 4. The product of an even integer times an integer with a factor 4 is an integer with a factor of 8.

OR: Since n is odd, $n = 2k+1$. Thus

$$n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4x = 4k(k+1)$$

and, since either k or $k{+}1$ is even, it follows that $k(k+1)$ is even and $n^2 - 1 = 4k(k+1)$ has a factor of 8.

b There are many solutions. Obviously, there must be one queen in each column and one in each row. The following comes from the observation that placing the queens in a "diagonal knight's move" pattern puts four non-attacking queens on the board immediately in the first four columns. If we start this pattern from the upper left corner, then there is no safe place for the queen on the bottom row, so we start it one down from the upper left corner. After these four are placed, mark the unsafe squares and find a pattern in the last four columns for the remaining four queens. In the following depiction of the chess board, `e` stands for a non-occupied square and `Q` for the placement of a queen.

```
eeeeeQee
Qeeeeeee
eeeeQeee
eQeeeeee
eeeeeeeQ
eeQeeeee
eeeeeeQe
eeeQeeee
```

c **641** The smallest possible sum is $1+2+3+4+6+7+8 = 36$, so none of the 35 positive integers less than 36 is possible. The largest possible sum is $81+82+83+84+85+86+87+88 = 676$. We now prove inductively that all of the positive integers not exceeding 676 except the first 35 are possible sums so that there are $676 - 35 = 641$ possible sums:

> Let $a_1, a_2, a_3, \ldots, a_8$ be a selection of 8 distinct positive integers not exceeding 88 with $a_i < a_{i+1}$ for $i = 1, 2, 3, \ldots, 7$. If the sum of these 8 numbers is less than 676, then either $a_i + 1 < a_{i+1}$ for some $i \le 7$ or $a_8 < 88$. In the first case replace a_i with $(a_i + 1)$ and in the second replace a_8 with $a_8 + 1$. In either case the new set will contain 8 distinct numbers not exceeding 88 and the sum will be one larger than the previous sum.

d For any 8-digit base-7 numeral $N = d_7d_6d_5d_4d_3d_2d_1d_0$, we have

$$\begin{aligned} N &= \sum_{m=0}^{7} d_m 7^m = \sum_{m=0}^{7} d_m (8-1)^m = \sum_{m=0}^{7} d_m \left(\sum_{k=0}^{m} \binom{m}{k} 8^{m-k}(-1)^k \right) \\ &= \sum_{m=0}^{7} d_m \left((-1)^m + \sum_{k=0}^{m-1} \binom{m}{k} 8^{m-k}(-1)^k \right) = \sum_{m=0}^{7} d_m \left((-1)^m + 8 \sum_{k=0}^{m-1} \binom{m}{k} 8^{m-k-1}(-1)^k \right) \\ &= \left(\sum_{m=0}^{7} (-1)^m d_m \right) + \left(\sum_{m=0}^{7} \left(8 \sum_{k=0}^{m-1} \binom{m}{k} 8^{m-k-1}(-1)^k \right) \right) \\ &= \left(\sum_{m=0}^{7} (-1)^m d_m \right) + 8 \left(\sum_{m=0}^{7} \left(\sum_{k=0}^{m-1} \binom{m}{k} 8^{m-k-1}(-1)^k \right) \right) = \left(\sum_{m=0}^{7} (-1)^m d_m \right) + 8A \end{aligned}$$

for A, an integer. If $d_7d_6d_5d_4d_3d_2d_1d_0$ is a palindrome, then $d_7d_6d_5d_4d_3d_2d_1d_0 = d_0d_1d_2d_3d_3d_2d_1d_0$ so

$$\left(\sum_{m=0}^{7} (-1)^m d_m \right) = (d_0 - d_1 + d_2 - d_3 + d_4 - d_5 + d_6 - d_7) = (d_0 - d_1 + d_2 - d_3 + d_3 - d_2 + d_1 - d_0) = 0$$

so $N = 8A$.

OR: For any 8-digit base-7 palindrome $N = d_0d_1d_2d_3d_3d_2d_1d_0$, we have

$$\begin{aligned} N &= d_0 7^7 + d_1 7^6 + d_2 7^5 + d_3 7^4 + d_3 7^3 + d_2 7^2 + d_1 7^1 + d_0 \\ &= d_0(7^7 + 1) + 7d_1(7^5 + 1) + 7^2 d_2(7^3 + 1) + 7^3 d_3(7^1 - 1). \end{aligned}$$

Since $(x+1)$ is a factor of $x^n + 1$ whenever n is a positive odd integer, $(7+1) = 8$ is a factor of the coefficient of each of the d_i, so 8 is a factor of N.

Problem 8: How Stupid Is Her Hamster?

a $\boxed{17/2 = 8.5}$ The possible exit choice sequences, their probability of being chosen, and their lengths are:

Seq :	C	AC	BC	ABC	BAC
Prob :	1/3	1/6	1/6	1/6	1/6
Len :	5	8	9	12	12

so the expected value is

$$\frac{1}{3} \cdot 5 + \frac{1}{6} \cdot 8 + \frac{1}{6} \cdot 9 + \frac{1}{6} \cdot 12 + \frac{1}{6} \cdot 12 = \frac{17}{2}.$$

b $\boxed{12}$ *The analysis analogous to* **a** *is more difficult since there are an infinite number of exit choice sequences. If one orders the sequences by the number of choices, then the infinite series summing the sums of the expected value for each number can be found. However, the following analysis uses only simple algebra:* Let e be the expected length of the trail every time the hamster is back in the cage. Then, if A is the first exit chosen, the expected length of the trail will be $3+e$ meters since the situation for the hamster after the run down tube A is the same as that before the run. Similarly, if the first exit is B, then the hamster can expect to run $4+e$ meters. Finally, if C is chosen, the hamster will run only 5 meters. Since the probability of choosing each of the three exits is $1/3$, we have

$$e = \frac{1}{3}(3+e) + \frac{1}{3}(4+e) + \frac{1}{3}(5) = 4 + \frac{2}{3}e$$

so $e = 12$.

c $\boxed{\frac{Rc}{P+Q+R} + \frac{P[R(a+c)+Q(a+b+c)]}{(P+Q+R)(Q+R)} + \frac{Q[R(b+c)+P(a+b+c)]}{(P+Q+R)(P+R)}}$ The possible exit choice sequences, their probability of being chosen, and their lengths are:

Seq :	C	AC	BC	ABC	BAC
Prob :	$\frac{R}{P+Q+R}$	$\left(\frac{PR}{(P+Q+R)(Q+R)}\right)$	$\left(\frac{QR}{(P+Q+R)(P+R)}\right)$	$\left(\frac{PQ}{(P+Q+R)(Q+R)}\right)$	$\left(\frac{QP}{(P+Q+R)(P+R)}\right)$
Len :	c	$a+c$	$b+c$	$a+b+c$	$b+a+c$

so the expected value is

$$\frac{Rc}{P+Q+R} + \frac{P[R(a+c)+Q(a+b+c)]}{(P+Q+R)(Q+R)} + \frac{Q[R(b+c)+P(a+b+c)]}{(P+Q+R)(P+R)}$$

d $\boxed{\frac{p(a-c)+q(b-c)+c}{1-p-q}}$ With an analysis analogous to **b** we have

$$e = p(a+e) + q(b+e) + (1-p-q)c = p(a-c) + q(b-c) + c + (p+q)e$$