1. **Fractionated**

Given the fractions $\frac{p}{q}$ and $\frac{r}{s}$ in lowest terms with p, q, r, and s positive integers, answer the following:

(3 pts.) a) What single fraction is halfway between the two fractions?

(6 pts.) b) Will the fraction $\frac{p+r}{q+s}$ always lie between the two fractions? If so, prove it. If not, give a counterexample.

(6 pts.) c) Suppose Shaq's free throw average (ratio of free throws made to those attempted) is better than Marcus's for *both* the months of December and January. Is it possible for Marcus's average to be better than Shaq's for the two months *combined*? If so, provide an example. If not, prove that it is impossible.

2. <u>Divide and Conquer</u>

(4 pts.) a) Assuming that a, b, c, and d are integers, show that the following sum of two fractions, each in lowest terms, cannot be an integer unless $b = d$.

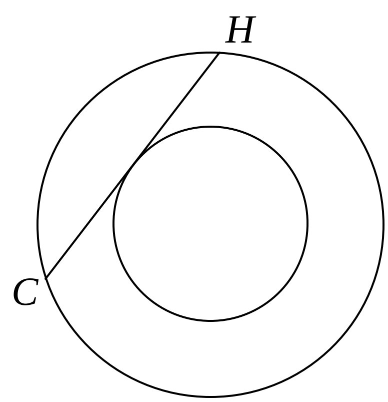
$$\frac{a}{b} + \frac{c}{d}$$

(5 pts.) b) The sum of two positive integers is 5,432 and their least common multiple is 223,020. Find the two integers.

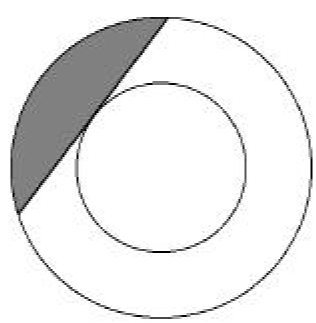
(6 pts.) c) Show that the sum $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ for $n > 1$ is never an integer.

3. **<u>Round and Round</u>**

(3 pts.) a) Let A be the annular region between two concentric circles. Chord $\overline{CH}$ of the larger circle is tangent to the smaller circle (see the diagram below). If $CH = 20$, find the number of square units in the annular region A.



(6 pts.) b) If the length of the chord in part **a** is L, and the diameter of the smaller circle is D, find a formula for the area of the segment (shaded area below) in terms of L and D.



(6 pts.) c) If $L = \sqrt{3}D$, and the area of the segment is $4p - 3\sqrt{3}$ square units, find the values for L and D.

4. **Progressively Better**

Let $a_1, a_2, a_3, \ldots a_n, \ldots$ be an arithmetic progression.

(3 pts.) a) Show that $\sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$.

(3 pts.) b) Suppose the terms of the progression are grouped as follows:

$\{a_1\}, \{a_2, a_3\}, \{a_4, a_5, a_6\}, \{a_7, a_8, a_9, a_{10}\}, \ldots$

If $a = a_1$, and d is the constant difference between consecutive terms, find a formula for the sum of the elements of the nth set in terms of a, d, and n.

(4 pts.) c) If n is a positive integer, show that 2^n cannot be written as the sum of two or more consecutive positive integers.

(5 pts.) d) Show that if c is a positive integer and $c \neq 2^n$ for any integer n, then c can be represented as the sum of two or more consecutive positive integers.

5. **Does it Really Make a Difference?**

Observe that some numbers can be expressed as a difference of two squares: $5 = 3^2 - 2^2$, $12 = 4^2 - 2^2$. Some numbers can be expressed as a difference of two squares in more than one way: $40 = 7^2 - 3^2 = 11^2 - 9^2$. Some numbers cannot be expressed as a difference of two squares: 10, 38.

(3 pts.) a) Express 28 and 48 as a difference of two squares in as many ways as each of them can be so represented.

(3 pts.) b) Show that 1170 cannot be expressed as a difference of two squares.

(4 pts.) c) Let *S* be the set of all positive integers that can be expressed as a difference of two squares in at least one way. Describe precisely those positive integers that are **not** in *S* and prove why they are not.

(5 pts.) d) For those numbers that are members of *S*, devise a formula for determining the number of ways that any one of them can be represented as a difference of two squares.

6. **<u>I’m lost and You R2.</u>**

You and your favorite robot R2D2 are both stranded on a lonely planet waiting for help to arrive. Being very tired you decide to take a nap but R2, who is never tired, slips out unnoticed to go for a walk. “He” does, however, leave you a note that you read several hours later. It informs you not where he is headed, but rather how he is going to get there. The note reads as follows:
“I shall proceed in a straight line for 10 miles. Then I shall alter my direction by 45° counterclockwise and travel 5 miles. Then I shall alter my direction by 45° counterclockwise and travel 2.5 miles. Then I shall alter my direction by 45° counterclockwise and travel 1.25 miles. Then I shall alter …” (You get the idea.)

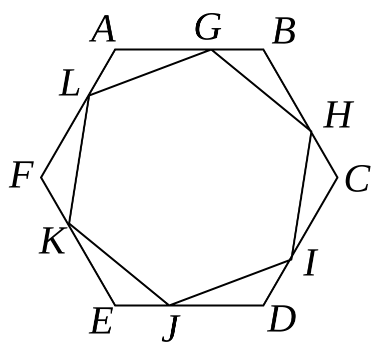
(4 pts.) a) Assume the terrain is superimposed with a Cartesian coordinate system with your location being the origin and R2’s initial travel direction is along the positive x axis. Also assume that the ground is sufficiently smooth and flat so as not to interfere with R2’s travel plans. What are the coordinates of the point that he is heading towards? If you give approximate coordinates, round off both to 3 decimal places.

(5 pts.) b) If R2 left two hours ago and travels at a constant rate of 4 miles per hour, then how many miles per hour do you have to average in order to arrive at the point in part **a** the same time as he? If you give an approximate speed, round off to 3 decimal places.

(6 pts.) c) Suppose R2 altered his direction by q degrees each time instead of 45°. Find a closed form expression in terms of q for the coordinates of his terminal position.
(You may use Euler’s identity, $e^{iq} = \cos q + i \sin q$ where $i = \sqrt{-1}$.)

7. **<u>What the Hex!</u>**

Consider a regular hexagon *ABCDEF* each of whose sides is 1 unit in length. Point *G* is on $\overline{AB}$ so that $GB = x$. Points *H*, *I*, *J*, *K*, and *L* are located consecutively on each of the hexagon's other five sides so that *GHIJKL* is also a regular hexagon (see the figure below).



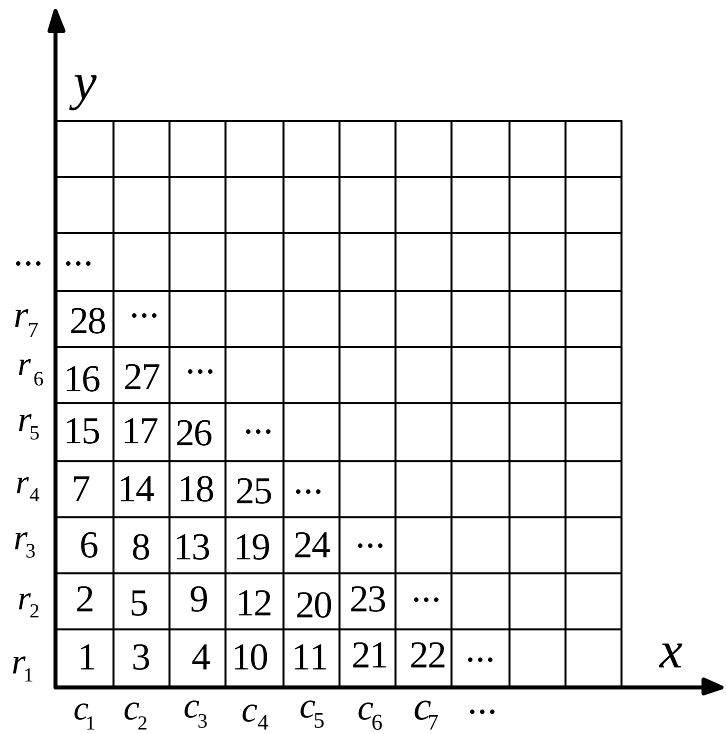
(4 pts.) a) Find a formula for the area of *GHIJKL* in terms of *x*.

(5 pts.) b) If the value of *x* is selected randomly from the interval $[0,1]$, then what is the probability that the area of *GHIJKL* is less than $\frac{31\sqrt{3}}{24}$ square units?

(6 pts.) c) If the probability that the area of *GHIJKL* is greater than *a* square units equals $\frac{1}{2}$, then find the value of *a*.

8. **<u>Lower Left Corner</u>**

Let the first quadrant of the Cartesian plane be partitioned into numbered "lattice cells" as shown in the diagram below. Assume the pattern of cell numbering continues on to all cells in the first quadrant. For example with the *n*th column and *n*th row of each cell designated by c_n and r_n respectively, cell 26 is located at (c_3, r_5), and the coordinates of the lower left hand corner of this cell are $(2,4)$.



(3 pts.) a) Find the coordinates of the lower left corner of cell 100.

(7 pts.) b) Given an arbitrary cell *N*, find a formula for the coordinates of the lower left hand corner of this cell.

(5 pts.) c) Determine <u>any and all</u> cells for which the graph of the curve $\{(x,y) \mid x^2 + y^2 = 442\}$ contains their lower left hand corners.