MAML Level 1 2000 Detailed Solutions

1. B) an integer

$\frac{n(n+1)}{2}$ is not positive if $n = 0$; $\frac{n(n+1)}{2}$ is an integer since either n or $n + 1$ must be even $\to$ their product is even making the expression an integer; $\frac{n(n+1)}{2}$ is not even if $n = 1$; $\frac{n(n+1)}{2}$ is not greater than n if $n = 0$; $\frac{n(n+1)}{2}$ is not odd if $n = 4$.

2. E) 9 and 11 only

$87^2 - 78^2 = 1485 \equiv 0 \bmod 99$; $593^2 - 395^2 = 195624 \equiv 0 \bmod 99$; $5681^2 - 1865^2 = 28795536 \equiv 0 \bmod 99$;
These and other examples seem to indicate the answer above. To prove the result in general, show
$\left(d_n 10^n + d_{n-1}10^{n-1} + ... + d_1 10^1 + d_0\right)^2 - \left(d_0 10^n + d_1 10^1 + ... + d_{n-1}10 + d_n\right)^2 \equiv 0 \bmod 99$; $10^2 \equiv 1 \bmod 99$;
When you expand the above expression some of the terms are of the form
$\left(d_i 10^i\right)^2 - \left(d_i 10^{n-i}\right)^2 = d_i^2\left(10^{2i} - 10^{2(n-i)}\right) \equiv d_i^2\left(1^i - 1^{(n-i)}\right) \bmod 99 \equiv 0 \bmod 99$;
The remaining terms are of the form
$2d_i 10^i d_j 10^j - 2d_i 10^{n-i} d_j 10^{n-j} = 2d_i d_j\left(10^{i+j} - 10^{2n-i-j}\right) = 2d_i d_j 10^{i+j}\left(1 - 10^{2n-2i-2j}\right) =$
$2d_i d_j 10^{i+j}\left(1 - 10^{2(n-i-j)}\right) \equiv 2d_i d_j 10^{i+j}\left(1 - 1^{(n-i-j)}\right) \bmod 99 \equiv 0 \bmod 99$; therefore the sum of all the terms in the expansion $\equiv 0 \bmod 99$.

3. D) 28

If the upper left hand corner date = x $\to$ upper right hand corner date = $x + 2$, the lower left hand corner date = $x + 14$ (2 weeks later), and the lower right hand corner date = $x + 16$;
$(x+2)(x+14) - x(x+16) = x^2 + 16x + 28 - x^2 - 16x = 28$.

4. E) 9:40

Clock F gains 6 minutes each of its "hours" on clock S. Since clock F is 60 minutes ahead of clock S, 10 of its hours have elapsed = 620 actual minutes. Subtracting 10 hours 20 minutes from 8:00 o'clock = 9:40.

5. B) $\frac{100(M-N)}{N}$

The fraction that M is greater than N = $\frac{M-N}{N} = \frac{100(M-N)}{N}\%$.

6. D) I or II only

Finding points on line ℓ, $\{(x,y)\,|\,3x+5y=15\}$ equidistant from the coordinate axes means points where $y=\pm x$. If $y=x \rightarrow 8x=15 \rightarrow x>0$ and $y>0$. If $y=-x \rightarrow -2x=15 \rightarrow x<0$ and $y>0$. This implies the points are in quadrants I or II.

7. D) 9

$$nw=36 \text{ and } (n+6)\left(w-\frac{1}{2}\right)=36 \rightarrow nw-\frac{1}{2}n+6w-3=36 \rightarrow 36-\frac{1}{2}n+6w-3=36 \rightarrow -n+12w-6=0$$
$$\rightarrow n=12w-6 \rightarrow w(12w-6)=36 \rightarrow 2w^2-w-6=0 \rightarrow (2w+3)(w-2)=0 \rightarrow$$
$$w=2 \rightarrow n=12(2)-6=18 \rightarrow \frac{n}{w}=9.$$

8. B) $-\frac{5}{4}$

The sum of the roots of an *n*th degree polynomial = $-\frac{a_{n-1}}{a_n}$ where a_n is the coefficient of x^n and a_{n-1} is the coefficient of x^{n-1}. Therefore the sum of the roots of the given polynomial = $-\frac{4}{8}=-\frac{1}{2}$. Since one of the roots = $\frac{1}{2}+i$ and the coefficients are all real, a second root must be its conjugate $\rightarrow$
$r+\frac{1}{2}+\frac{1}{2}+i+\frac{1}{2}-i=-\frac{1}{2} \rightarrow r=-2$ where *r* is the fourth root of the polynomial $\rightarrow$
the product of all the roots = $(-2)\left(\frac{1}{2}\right)\left(\frac{1}{2}+i\right)\left(\frac{1}{2}-i\right)=-\frac{5}{4}$.

9. E) $\frac{4p}{45}$

Since $\sin\left(\frac{1}{2}q\right)=\cos(2q+50°) \rightarrow \frac{1}{2}q+2q+50°=90° \rightarrow \frac{5}{2}q=40° \rightarrow q=16° \rightarrow q=16\left(\frac{p}{180}\right)=\frac{4p}{45}$.

10. C) 9^{x+1}

$$n=3^x+3^x+3^x=3\cdot 3^x=3^{x+1} \rightarrow n^2=\left(3^{x+1}\right)^2=\left(3^2\right)^{x+1}=9^{x+1}.$$

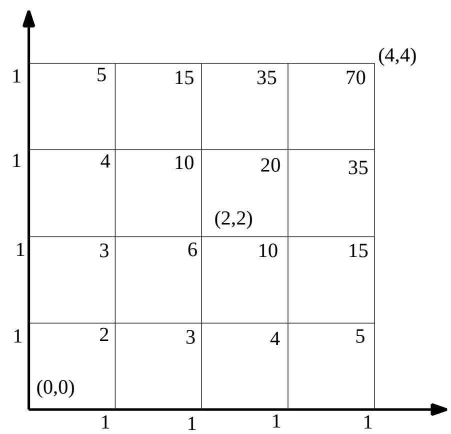
11. B) $\sqrt{b^2-3}$

Let $BD=x$. $1^2+b^2=AB^2=2^2+x^2 \rightarrow x^2=b^2-3 \rightarrow x=\sqrt{b^2-3}$.

12. C) $\dfrac{2\left(a^2+1\right)}{(a+1)^2}$

Since $\tan\left(x+\dfrac{\pi}{4}\right)=a \rightarrow \dfrac{\tan x+1}{1-\tan x}=a \rightarrow \tan x+1=a-a\tan x \rightarrow a\tan x+\tan x=a-1 \rightarrow$
$\tan x(a+1)=a-1 \rightarrow \tan x=\dfrac{a-1}{a+1}$; $\sec^2 x=1+\tan^2 x=1+\left(\dfrac{a-1}{a+1}\right)^2=\dfrac{(a+1)^2+(a-1)^2}{(a+1)^2}=\dfrac{2\left(a^2+1\right)}{(a+1)^2}$.

13. E) $\dfrac{18}{35}$



Under the given conditions, the number of paths to a point is arrived at by adding the numbers on the vertices leading to this point. There are 70 paths from $(0,0)$ to $(4,4)$. There are 6 paths from $(0,0)$ to $(2,2)$ and therefore 6 paths starting at $(2,2)$ to $(4,4)$. This means there are 36 (6×6) paths from $(0,0)$ to $(2,2)$ to $(4,4)$.
The probability that the pin starting at $(0,0)$ ending at $(4,4)$ goes through $(2,2)$ = $\dfrac{36}{70}=\dfrac{18}{35}$.

14. D) $2a^2$

The inequality $|x-1|+|y-2|\le a$ is a translation 1 unit to the right and 2 units up of the inequality $|x|+|y|\le a$. Therefore the two inequalities enclose the same area. $|x|+|y|\le a$ encloses the area of the square whose vertices are $(0,\pm a)$ and $(\pm a,0)$. The area of the square is one-half the product of its diagonals = $\left(\dfrac{1}{2}\right)(2a)(2a)=2a^2$.

15. C) 6

Let A = unshaded area in the diagram. The lightly shaded area = $4\pi+4\pi+\pi-A=9\pi-A$.
The darkly shaded area = πr^2-A where r is the radius of the darkly shaded circle.
Therefore $\pi r^2-A=9\pi-A \rightarrow r=3$ and the diameter of the darkly shaded circle = 6.

16. D) 1.44π

If r = radius of the inscribed circle, A = area of a triangle, and P = perimeter of a triangle, then
$\dfrac{1}{2}P\cdot r=A \rightarrow 10r=12 \rightarrow r=1.2 \rightarrow \text{area of inscribed circle}=1.44\pi$.

17. D) 9π

Use the sum to product formula $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ to simplify the equation

$\sin(5x)+\sin(x)=\sin(3x) \to 2\sin(3x)\cos(2x)-\sin(3x)=0 \to \sin(3x)(2\cos(2x)-1)=0 \to$

$\sin(3x)=0$ or $\cos(2x)=\frac{1}{2} \to 3x=0, p, 2p, 3p, 4p, 5p$ or $2x=\frac{p}{3}, \frac{5p}{3}, \frac{7p}{3}, \frac{11p}{3} \to$ sum of all the solutions for $x = \frac{15p}{3}+\frac{24p}{6}=9p$.

18. D) $\frac{9}{2}$

$\log_6 54+\log_x 16=\log_{\sqrt{2}} x-\log_{36}\left(4/9\right) \to \frac{\log 54}{\log 6}+\frac{\log 16}{\log x}=\frac{\log x}{\log\sqrt{2}}-\frac{\log\left(4/9\right)}{\log 36} \to$

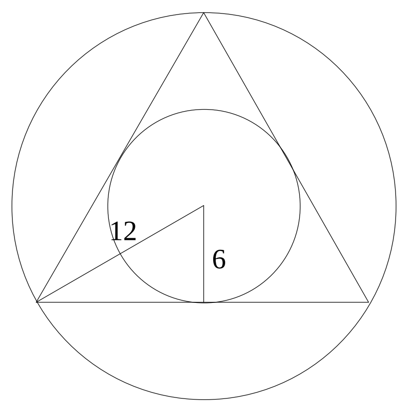
$\frac{\log 54}{\log 6}+\frac{4\log 2}{\log x}=\frac{2\log x}{\log 2}-\frac{2\log\left(2/3\right)}{2\log 6} \to \frac{\log 54}{\log 6}+\frac{\log\left(2/3\right)}{\log 6}+\frac{4\log 2}{\log x}-\frac{2\log x}{\log 2}=0 \to$

$\frac{\log 36}{\log 6}+\frac{4\log 2}{\log x}-\frac{2\log x}{\log 2}=0 \to 2+\frac{4}{y}-2y=0$ if $y=\frac{\log x}{\log 2} \to$

$-2y^2+2y+4=0 \to y^2-y-2=0 \to (y+1)(y-2)=0 \to \frac{\log x}{\log 2}=-1 \text{ or } 2 \to \log_2 x=-1 \text{ or } 2 \to$

$x=\frac{1}{2}$ or $4 \to$ sum of the values for $x = \frac{9}{2}$.

19. C) 36π



Since the circumference of the circumscribed circle = 24π, the radius of the circumscribed circle = 12. The right triangle formed by the circumscribed radius, the inscribed radius and half the side of the equilateral triangle is a 30-60-90° triangle. Therefore the radius of the inscribed circle = 6 and its area = 36π.

20. E) $\frac{p+2q}{3}$

Let $a_1 = \frac{p+q}{2}$; $a_2=\frac{\frac{p+q}{2}+q}{2}=\frac{p+3q}{4}$; $a_3=\frac{\frac{p+q}{2}+\frac{p+3q}{4}}{2}=\frac{3p+5q}{8}$;

$a_4=\frac{\frac{3p+5q}{8}+\frac{p+3q}{4}}{2}=\frac{5p+11q}{16}$; $a_5 = \frac{\frac{3p+5q}{8}+\frac{5p+11q}{16}}{2}=\frac{11p+21q}{32}$;

the previous results establishes the following pattern for a_n : the coefficient of p = $\dfrac{\frac{1}{3}\left(2^n+(-1)^{n+1}\right)}{2^n}$;

the coefficients of p and q add to 1$\rightarrow$ coefficient of q = $\dfrac{\frac{1}{3}\left(2^{n+1}-(-1)^{n+1}\right)}{2^n}$;

as $n\to\infty$, the coefficient of $p\to\dfrac{1}{3}$ and so the coefficient of $q\to\dfrac{2}{3}$;

therefore as $n\to\infty$, $a_n\to\dfrac{p+2q}{3}$; to prove the above patterns use mathematical induction:

the formulas are true for $n=1$ and 2, now show than $a_{n+1}=\dfrac{a_{n-1}+a_n}{2}$:

$$\frac{a_{n-1}+a_n}{2}=\frac{\dfrac{\frac{1}{3}\left(2^{n-1}+(-1)^n\right)}{2^{n-1}}p+\dfrac{\frac{1}{3}\left(2^n-(-1)^n\right)}{2^{n-1}}q+\dfrac{\frac{1}{3}\left(2^n+(-1)^{n+1}\right)}{2^n}p+\dfrac{\frac{1}{3}\left(2^{n+1}-(-1)^{n+1}\right)}{2^n}q}{2}=$$

$$\frac{\frac{1}{3}\left(2^{n-1}+(-1)^n\right)}{2^n}p+\frac{\frac{1}{3}\left(2^n-(-1)^n\right)}{2^n}q+\frac{\frac{1}{3}\left(2^n+(-1)^{n+1}\right)}{2^{n+1}}p+\frac{\frac{1}{3}\left(2^{n+1}-(-1)^{n+1}\right)}{2^{n+1}}q=$$

$$\frac{\frac{1}{3}\left(2^n+2(-1)^n\right)}{2^{n+1}}p+\frac{\frac{1}{3}\left(2^{n+1}-2(-1)^n\right)}{2^{n+1}}q+\frac{\frac{1}{3}\left(2^n+(-1)^{n+1}\right)}{2^{n+1}}p+\frac{\frac{1}{3}\left(2^{n+1}-(-1)^{n+1}\right)}{2^{n+1}}q=$$

$$\frac{\frac{1}{3}\left(2\cdot2^n+2(-1)^n+(-1)^{n+1}\right)}{2^{n+1}}p+\frac{\frac{1}{3}\left(2\cdot2^{n+1}-2(-1)^n+(-1)^{n+1}\right)}{2^{n+1}}q=$$

$$\frac{\frac{1}{3}\left(2^{n+1}+(-1)^n\right)}{2^{n+1}}p+\frac{\frac{1}{3}\left(2^{n+2}-(-1)^n\right)}{2^{n+1}}q=\frac{\frac{1}{3}\left(2^{n+1}+(-1)^{n+2}\right)}{2^{n+1}}p+\frac{\frac{1}{3}\left(2^{n+2}-(-1)^{n+2}\right)}{2^{n+1}}q=a_{n+1}.$$

21. D) $\dfrac{19}{8}$

$$\sum_{i=1}^{10}s_i=\frac{10}{2}\left(2s_1+9\left(s_2-s_1\right)\right)=\frac{10}{2}\left(2t_1+9\left(2t_2-t_1\right)\right)=5\left(18t_2-7t_1\right)=90t_2-35t_1;$$

$$\sum_{i=1}^{15}t_i=\frac{15}{2}\left(2t_1+14\left(t_2-t_1\right)\right)=\frac{15}{2}\left(14t_2-12t_1\right)=105t_2-90t_1;$$

$$105t_2-90t_1=90t_2-35t_1\to15t_2=55t_1\to t_2=\frac{11}{3}t_1;\ \frac{s_2-s_1}{t_2-t_1}=\frac{2t_2-t_1}{t_2-t_1}=\frac{2\cdot\frac{11}{3}t_1-t_1}{\frac{11}{3}t_1-t_1}=\frac{\frac{19}{3}t_1}{\frac{8}{3}t_1}=\frac{19}{8}.$$

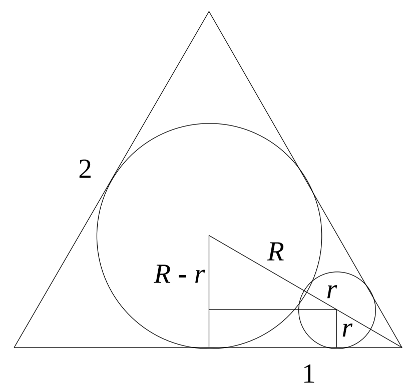
22. A) 0

Since $\left(r+\frac{1}{r}\right)^2=3\rightarrow r^2+2+\frac{1}{r^2}=3\rightarrow r^2+\frac{1}{r^2}=1;\ r^3+\frac{1}{r^3}=\left(r+\frac{1}{r}\right)\left(r^2-1+\frac{1}{r^2}\right)=\left(r+\frac{1}{r}\right)(1-1)=0$.

23. E) 63825

There are $5 \times 5 \times 5 = 125$ different numbers that can be formed from the digits 2, 3, 5, 6, and 7, with digits appearing more than once allowed. Therefore if the numbers are listed in a column, each digit appears 25 times in each column $\rightarrow$ sum of each column = $25(2+3+5+6+7)=575$ $\rightarrow$ units digit in the sum = 5; 575 + 57 = 632 $\rightarrow$ ten's digit = 2; 575 + 63 = 638; the sum = 63825.

24. E) $\frac{11p}{24}$



If R and r are the radii of the first 2 circles, the right triangle in the diagram has a hypotenuse of length $R+r$ and a leg of length $R-r$. This is a 30-60-90° triangle. Therefore $R+r=2(R-r)$ $\rightarrow$ $R=3r$. This relationship must also hold as each set of smaller circles is inscribed in the triangle and tangent to the circles. The radius of the inscribed circle = $\frac{1}{\sqrt{3}}$ $\rightarrow$ its area = $\frac{p}{3}$; the next 3 circles have a radius which equals $\frac{1}{3\sqrt{3}}$ $\rightarrow$ their total area = $3p\left(\frac{1}{3\sqrt{3}}\right)^2=\frac{p}{9}$. The next three circles after this have a radius which equals $\frac{1}{3}$ the previous radius $\rightarrow$ their total area = $\frac{1}{9}$ the previous area. An infinite geometric series is formed with a first term equaling $\frac{p}{9}$ and a ratio between terms equaling $\frac{1}{9}$. Therefore the total area of all the circles formed as this process continues forever = $\frac{p}{3}+\frac{\frac{p}{9}}{1-\frac{1}{9}}=\frac{p}{3}+\frac{p}{8}=\frac{11p}{24}$.

25. A) 25%

For the cone, $h=3d=6r\rightarrow V=\frac{1}{3}pr^2(6r)=2pr^3$; the volume of the 2 spherical scoops =

$2\cdot\frac{4}{3}pr^3=\frac{8}{3}pr^3$; the excess = $\frac{2}{3}pr^3$; the part of the scoops outside the cone = $\frac{\frac{2}{3}pr^3}{\frac{8}{3}pr^3}=\frac{1}{4}=25\%$.

26. B) $\frac{3}{2}\sqrt{26}$

The area of $\Delta ABC = \frac{1}{2} AB \cdot AC \sin A; AB = \sqrt{(0-1)^2 + (2-3)^2 + (5-0)^2} = \sqrt{27} = 3\sqrt{3}$;

$AC = \sqrt{(-1-1)^2 + (0-3)^2 + (2-0)^2} = \sqrt{17}; \overrightarrow{AB} = -1i - 1j + 5k; \overrightarrow{AC} = -2i - 3j + 2k$;

$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\| \left\|\overrightarrow{AC}\right\|} = \frac{2+3+10}{3\sqrt{3}\sqrt{17}} = \frac{5}{\sqrt{51}} \to \sin A = \frac{\sqrt{26}}{\sqrt{51}}$; area of $\Delta ABC = \frac{1}{2} \cdot 3\sqrt{3}\sqrt{17}\frac{\sqrt{26}}{\sqrt{51}} = \frac{3}{2}\sqrt{26}$.

27. E) $\frac{\sqrt{2-x^2}}{2-x^2}$

Let $a = \cos^{-1}(x) \to 0 \le a \le p$ and $\cos(a) = x \to \sin(a) = \sqrt{1-x^2}$.

Let $b = \tan^{-1}\left(\sqrt{1-x^2}\right) \to 0 \le b < \frac{p}{2}$ and $\tan(b) = \sqrt{1-x^2} \to$

$\cos(b) = \frac{1}{\sqrt{1^2 + \left(\sqrt{1-x^2}\right)^2}} = \frac{1}{\sqrt{2-x^2}} = \frac{\sqrt{2-x^2}}{2-x^2}$.

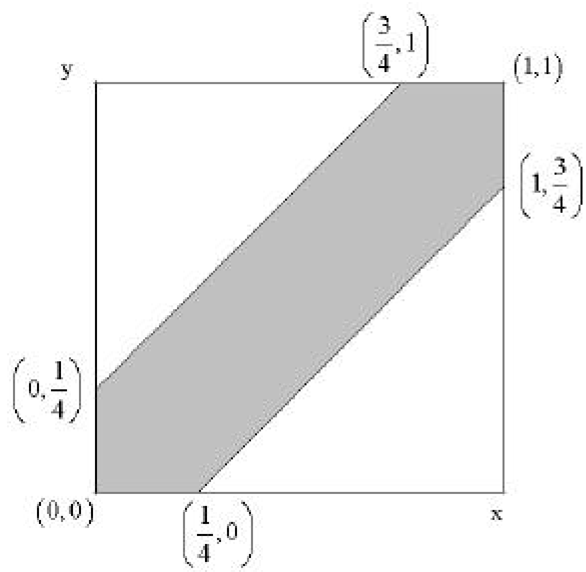
28. B) $\frac{7}{16}$

Let x and y represent the coordinates of the two points selected on the interval $[0,1]$ Since the distance between them is less than one-fourth, they must satisfy the inequality $|x-y| < \frac{1}{4}$.

Let the sample space be the unit square shown on the right. The successful event is the area satisfying the inequality which is the shaded region between the lines $x - y = \frac{1}{4}$ and $x - y = -\frac{1}{4}$.

The shaded area = area of the square – area of the 2 right isosceles triangles =

$1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$.



29. C) 32

Generating 3 more rows, a pattern is established. Row 2^n consists of all 1's. Row $2^n + 1$ consists of all 0's between the end 1's. Row 64 consists of 64 1's and row 65 has 63 0's between the end 1's. Row 96 is 32 rows past row 64 and would consist of 32 1's followed by 32 0's followed by 32 1's.

Row 1									1								
Row 2								1		1							
Row 3							1		0		1						
Row 4						1		1		1		1					
Row 5					1		0		0		0		1				
Row 6				1		1		0		0		1		1			
Row 7			1		0		1		0		1		0		1		
Row 8		1		1		1		1		1		1		1		1	
Row 9	1		0		0		0		0		0		0		0		1

Therefore $g(96) - f(96) = 64 - 32 = 32$. Note as more rows are generated, the same triangular arrays appear imbedded within the triangle. Hence the rows from 1 to 32 appear twice from row 65 to row 96. A similar occurrence is rows 1 to 4 appearing twice from rows 5 to 8.

30. D) 850

The center of the circles must lie on the perpendicular bisector of the segment connecting $A(1,9)$ and $B(8,8)$. The midpoint of $\overline{AB} = (4.5, 8.5)$.

The slope of this line = $-\left(\dfrac{8-9}{8-1}\right)^{-1} = 7 \rightarrow$ line is $y - 8.5 = 7(x - 4.5) \rightarrow y = 7x - 23$.

The distance from any point on this line to the x-axis = $|7x - 23|$. The distance from any point on this line to $(1,9)$ is $\sqrt{(x-1)^2 + (7x-32)^2}$. Setting these distances equal produces the equation:

$(7x-23)^2 = (x-1)^2 + (7x-32)^2 \rightarrow x^2 - 128x + 496 = 0 \rightarrow (x-4)(x-124) = 0 \rightarrow x = 4, 124 \rightarrow$ radii are $7(4) - 23 = 5$ and $7(124) - 23 = 845 \rightarrow$ sum of the radii = 850.