THIRTY-FIFTH ANNUAL OLYMPIAD HIGH SCHOOL PRIZE COMPETITION IN MATHEMATICS

1998-1999

Conducted By
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Of
Mathematics Leagues
(MAML)

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The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

TUESDAY, OCTOBER 20, 1998

1. Assume that of all the students taking this test, 95% get this question correct while only 8% get question #30 correct. If the relationship between the percent of students getting a question correct and the question number is linear, what percent should get question #23 correct?

A) 25% B) 27% C) 29% D) 31% E) 35%

2. The ratio of the measures of the angles of a triangle is 3 : 4 : 8. When the measure of the largest interior angle is subtracted from the measure of the largest exterior angle, the result is k degrees. Determine the value of k.

A) 12 B) 36 C) 48 D) 84 E) 96

3. Fermat was born in the year $(7x + 12)$ and was 17 in the year $(8x - 198)$. In what year was his age $\frac{x-7}{22}$ years old?

A) 1584 B) 1601 C) 1611 D) 1618 E) 1635

4. There is only one set of four consecutive integers where the sum of the cubes of the smallest three equals the cube of the fourth. What is the fourth integer?

A) 3 B) 5 C) 6 D) 7 E) 9

5. A circle is inscribed in equilateral triangle ABC and ΔABC is inscribed in a circle of radius R. Determine, in terms of R, the area of the region bounded by the two circles.

A) $\frac{1}{2}\pi R^2$ B) $2\pi R^2$ C) $\frac{1}{3}\pi R^2$ D) $3\pi R^2$ E) $\frac{3}{4}\pi R^2$

6. If $P(x) = 6x^2 + Ax + 6$, it can be factored over the positive integers into exactly two binomials in several different ways. If P(x) is factored, what is the probability that A is a prime number?

A) $\frac{1}{8}$ B) $\frac{1}{7}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{2}{7}$

7. If $a_0 = 1, a_1 = 3$, and for $n \geq 1$, $(a_n)^2 - (a_{n-1})(a_{n+1}) = (-1)^n$, then determine a_4.

A) 33 B) $33\frac{2}{3}$ C) $108\frac{4}{5}$ D) 109 E) 360

8. If each person made one true and one false statement, what is the correct order in which they spoke?

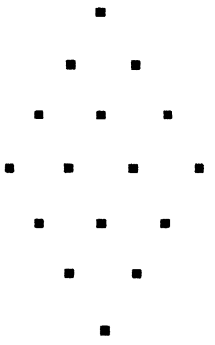
i) A: I was first. B was second.
ii) B: I was third. A was last.
iii) C: I was second. D was fourth.
iv) D: I was fourth. E was first.
v) E: I was third. C was second.

A) ACEBD B) CBDEA C) CBADE D) CBEDA E) ACBED

9. A club identifies each of its members with a unique three-digit number, the sum of whose digits is 20. What is the maximum number of members possible for this club?

A) 24 B) 27 C) 30 D) 33 E) 36

10. Shown at the right is an array of 16 points. What is the maximum number of equilateral triangles that could be formed by selecting triples of points from the array?



A) 24 B) 28 C) 30 D) 32 E) 36

11. The lengths of the sides of a scalene triangle are one-digit prime numbers. The bisector of the largest angle divides the opposite side into segments whose positive difference is:

A) $\frac{3}{4}$ B) 1 C) $\frac{5}{4}$ D) $\frac{3}{2}$ E) $\frac{7}{4}$

12. If $x^2 - y^2 = 1331$ for positive integers x and y, find the minimum value of xy.

A) 990 B) 3131 C) 3630 D) 3990 E) 442,890

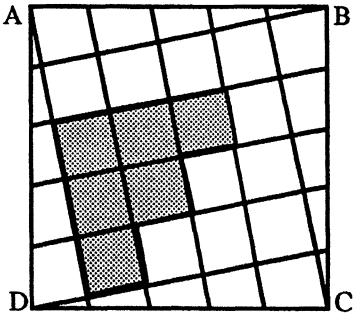
13. In ΔPQR, $m\angle R = 90°$, $QR = 2$, and S lies on $\overline{PR}$ so that $RS = SP = 1$. Then $\tan \angle PQS$ can be written as:

A) $\frac{1}{2}$ B) $\frac{1}{\sqrt{5}}$ C) $\tan 22.5°$ D) $\frac{1}{3}$ E) $\frac{1}{6}$

14. At the start of a card game A, B, and C had money in the ratio 7 : 6 : 5 respectively. After the game the ratio was 6 : 5 : 4. If one of them won 9 dollars, how much did B have at the start?

A) \$45 B) \$54 C) \$216 D) \$270 E) \$315

15. Each side of square ABCD is divided into five equal parts. Determine the ratio of the area of the shaded region to the area of ABCD.



A) $\frac{3}{16}$ B) $\frac{1}{5}$ C) $\frac{3}{13}$ D) $\frac{6}{25}$ E) $\frac{1}{4}$

16. Four spheres, each of radius 10 inches, lie on a horizontal table so that the centers of the spheres form a square of side 20 inches. A fifth sphere of radius 10 inches is tangent to all four spheres. How many inches above the table is the center of the fifth sphere?

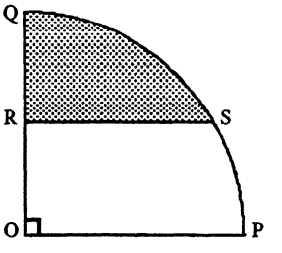
A) $10\sqrt{6}$ B) $10 + 10\sqrt{2}$ C) $10 + 10\sqrt{3}$ D) $40 - 10\sqrt{2}$ E) 24

17. A point P(x, y) moves so that the sum of the distances from P to the coordinate axes is equal to the distance from P to the point A(1, 1). Determine the equation of the locus of P in the first quadrant.

A) $(x + 1)(y + 1) = 1$ B) $(x + 1)(y + 1) = 2$ C) $(x - 1)(y - 1) = 1$

D) $(x - 1)(y - 1) = 2$ E) $xy = 1$

18. Arc QSP is a quarter circle, R is the midpoint of $\overline{OQ}$, and $\overline{RS} \parallel \overline{OP}$. Which of the following is closest to the ratio of the shaded area to the area of the quarter circle?



A) .33 B) .36 C) .39 D) .42 E) .45

19. Find all real values of x which make the following statement true:

$$\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$$

A) $\frac{1}{16}$ B) $\frac{1}{4}$ C) $\frac{1}{4}, \frac{1}{16}$ D) $\frac{1}{8}$ E) $\frac{1}{2}$

20. Define a shuffling function f by $f(A, B, C, D) = (P, Q, R, S)$ where (P, Q, R, S) is a permutation of (A, B, C, D). Function f produces the same transformation each time it is applied. If $f(f(f(A, B, C, D))) = (C, D, B, A)$, then $f(A, B, C, D)$ equals:

A) (B, C, D, A) B) (D, A, B, C) C) (D, C, A, B)

D) (C, D, B, A) E) (C, B, D, A)

21. All the jacks, queens, kings, and aces of a regular 52 card deck are chosen. The 16 cards are thoroughly shuffled and my opponent, a person who always tells the truth, simultaneously draws two cards at random. She says, "I hold at least one ace." Determine the probability that she holds two aces.

A) $\frac{1}{5}$ B) $\frac{3}{16}$ C) $\frac{1}{6}$ D) $\frac{2}{15}$ E) $\frac{1}{9}$

22. Each of four students chooses at random one integer from 1 to 10 inclusive. What is the probability that at least two of them choose the same integer?

A) .400 B) .450 C) .496 D) .500 E) .504

23. A rectangular billiard table has vertices at P(0, 0), Q(0, 7), R(10, 7), and S(10, 0). A small billiard ball starts at M(3, 4) and moves in a straight line to the top of the table, bounces to the right side of the table, then comes to rest at N(7, 1). What is the y-coordinate of the point where it hits the right side?

A) 3.7 B) 3.8 C) 3.9 D) 4 E) 4.1

24. In terms of a what is the sum of the squares of the reciprocals of the roots of the equation $ax^2 + x + a = 0$?

A) $\frac{1}{a} - 4$ B) $\frac{1}{a^2} - 4$ C) $\frac{1}{a^6} - 4$ D) $\frac{1}{a^4} - 2$ E) $\frac{1}{a^2} - 2$

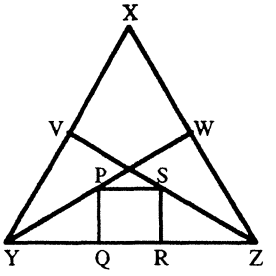
25. If $f\left(\frac{y}{y-1}\right) = \frac{1}{y}$ for all $y \neq 0, 1$, and $0 < x < \frac{\pi}{2}$, then an expression for $f(\csc^2 x)$ would be:

A) $\sin^2 x$ B) $\cos^2 x$ C) $\tan^2 x$ D) $\cot^2 x$ E) $\sec^2 x$

26. Determine the sum: $\frac{75}{90} + \frac{75}{110} + \frac{75}{132} + \frac{75}{156} + \frac{75}{182} + \ldots + \frac{75}{9900}$.

A) 7.5 B) $7.5\overline{3}$ C) $7.5\overline{6}$ D) $7.58\overline{3}$ E) $7.58\overline{6}$

27. Each side of ΔXYZ has a length of 3. If $\overline{ZV}$ and $\overline{YW}$ are medians and PQRS is a square, determine PQ.



A) $\frac{3\sqrt{3}-3}{11}$ B) $\frac{3\sqrt{3}+3}{11}$ C) $\frac{3\sqrt{3}+6}{11}$ D) $\frac{6\sqrt{3}-3}{11}$ E) $\frac{6\sqrt{3}+3}{11}$

28. If the equation $(x^2 - 4)(x^2 - 1) = k$ has four non-zero, real roots equally spaced on the number line, and if x_4 is the largest of those roots, then determine the product kx_4.

A) $\frac{7}{8}$ B) $\frac{7}{4}$ C) $\frac{7\sqrt{2}}{8}$ D) $\frac{7\sqrt{2}}{4}$ E) $\frac{21\sqrt{2}}{8}$

29. Four pairs of socks are placed side-by-side on a straight clothes line. The socks in each pair are identical, but the pairs themselves are of different colors. If I stand on one side of the line, how many different color arrangements can I see if no sock is allowed to be next to its mate?

A) 792 B) 630 C) 2520 D) 864 E) 720

30. A hexagon is inscribed in a circle. Moving clockwise the lengths of the sides are 1, 1, 1, 2, 2, and 2. What is the hexagon's area?

A) $3\sqrt{3}$ B) $\frac{13\sqrt{3}}{4}$ C) $\frac{7\sqrt{3}}{2}$ D) $\frac{15\sqrt{3}}{4}$ E) $4\sqrt{3}$