Problem One: Taking the Shortcut (14 pts)

The following examples indicate that certain pairs of two-digit numbers can be multiplied in a quick and easy way:

$47 \cdot 43 = 2021$ $\quad 62 \cdot 68 = 4216$ $\quad 74 \cdot 76 = 5624$

(3 pts) a) Using the method suggested by the examples, write down the products of $79 \cdot 71$ and $35 \cdot 35$.

(4 pts) b) Describe the following:
i) the set of all pairs of two-digit numbers for which the method works.
ii) the shortcut method used to find their product.

(7 pts) c) Prove, in general, that the method employed in (a) is valid for the pairs of numbers described in (bi).

Problem Two: Growing by Leaps and Bounds (10 pts)

(4 pts) a) Shown below is a sequence of areas of rectangles whose sides are integers. Determine the area of the 10th rectangle in the sequence.

$$1, 2, 6, 15, 40, 104, \ldots$$

(6 pts) b) Find the sum of the series: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{999000}$.

Problem Three: Divide and Conquer (14 pts)

(6 pts) a) Prove that $2^3 - 1$ divides $2^{24} - 1$.

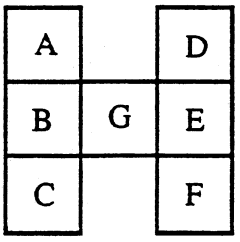
(8 pts) b) If a divides b, prove that $2^a - 1$ divides $2^b - 1$.

Massachusetts Olympiad Level II – 1998

Problem Four: Hopscotch Mathematics (15 pts)

(6 pts) a) In the diagram below each of the seven digits chosen from 0 through 9 corresponds to a different letter and the following products are equal:

$$A \cdot B \cdot C = D \cdot E \cdot F = B \cdot G \cdot E$$



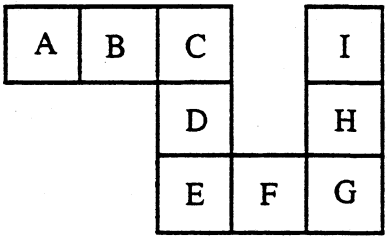
i) Determine the product of those digits which are not used.

ii) Determine the value of G.

iii) Determine the number of distinct arrangements of the digits.

(9 pts) b) In the diagram below each of the digits 1 through 9 corresponds to a different letter and the following sums are equal:

$$A + B + C = C + D + E = E + F + G = G + H + I$$



i) If $A + B + C = 17$ then find the values of

a) $C + E + G$ b) E c) $D + F$

ii) If $A + B + C = 16$, then find all values for E.

iii) Determine, with reasons, the minimum value of $A + B + C$ and show an arrangement of the digits using that value of $A + B + C$.

Problem Five: Function Pie (18 points):

Given $A(-\frac{1}{2}, 0)$ and $B(\frac{1}{2}, 0)$, let P(x, y) be any point in the plane such that $y > 0$. Let F be a function of P such that $F(P)$ is the measure of $\angle APB$ in radians.

(3 pts) a) For point $P(-\frac{1}{2}, \sqrt{3})$, determine $F(P)$.

(3 pts) b) If P(0, y) lies on the y-axis, find $F(P)$ in terms of y.

(3 pts) c) Given P as in (b), find the range of $F(P)$.

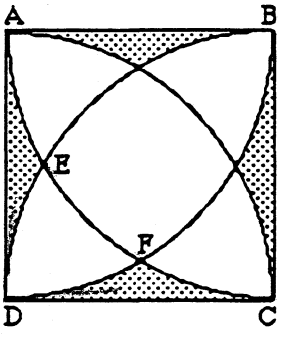
(3 pts) d) Find the equation of all points P(x, y) such that $F(P) = \frac{\pi}{4}$.

(3 pts) e) Let P lie on the line $x = k$ where $k > \frac{1}{2}$. Prove that $F(P) < \frac{\pi}{2}$.

(3 pts) f) Determine the largest region R such that all points P in R satisfy $F(P) > \frac{\pi}{2}$. Justify your answer.

Problem Six: Only the Shadow Knows (15 pts)

The diagram shows a square of side 12 where A, B, C, and D are the centers of circles with radii of length 12.



(5 pts) a) Determine the exact area of the shaded region. Show your work

(5 pts) b) Determine the exact length of line segment $\overline{EF}$. Explain your reasoning.

(5 pts) c) Determine the exact value of $\sin\angle EDF$. Explain your reasoning.

Massachusetts Olympiad Level II – 1998

Problem Seven: Back to the Future — Problem 25, October 21, 1997 (14 pts)

(6 pts) a) Triangle ABC is equilateral with $\frac{AD}{DC} = \frac{CF}{FB} = \frac{BE}{EA} = \frac{1}{2}$. Find the ratio of the area of ΔPQR to the area of ΔABC.

(8 pts) b) Triangle ABC is equilateral with $\frac{AD}{DC} = \frac{CF}{FB} = \frac{BE}{EA} = \frac{1}{n-1}$. Find a formula in terms of n for the ratio of the area of ΔPQR to the area of ΔABC.