

**THIRTY-FOURTH ANNUAL OLYMPIAD**

**HIGH SCHOOL**

**PRIZE COMPETITION IN MATHEMATICS**

**1997-1998**

**Conducted by  
The Massachusetts Association  
of  
Mathematics Leagues  
(MAML)**

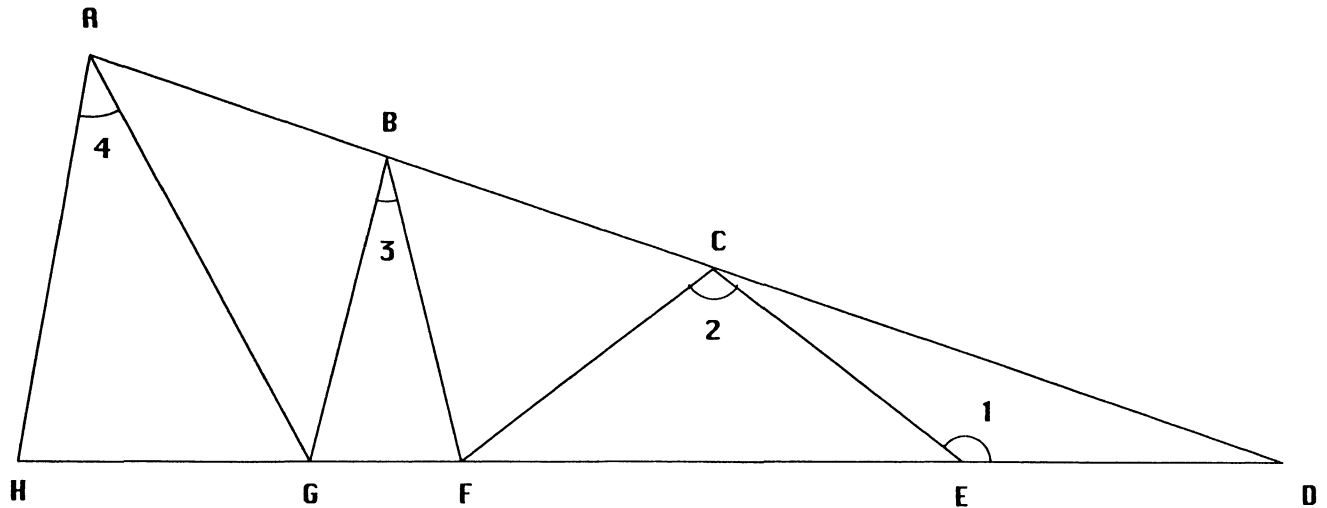
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**FIRST LEVEL EXAMINATION**

**TUESDAY, OCTOBER 21, 1997**

Please note: Diagrams are not necessarily drawn to scale.

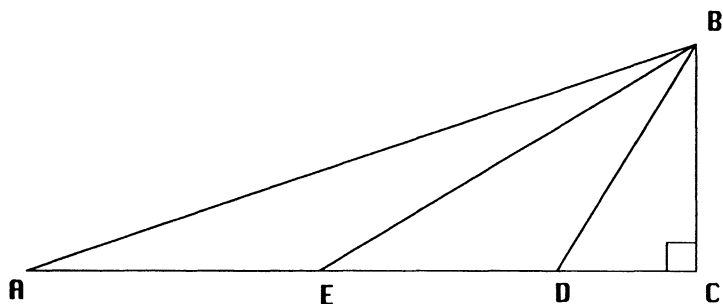
1.



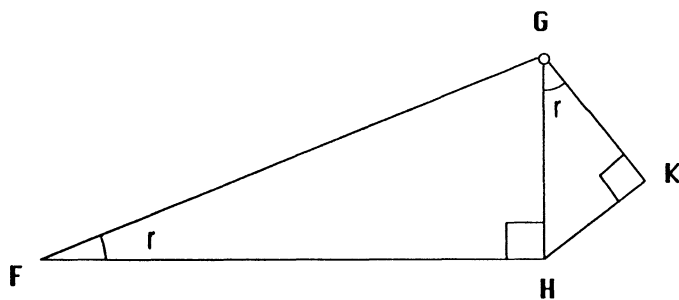
In  $\triangle ADH$ ,  $AD = DH$ ,  $DE = EC = CF = FB = BG = GA = AH$ .  $\angle 1 + \angle 2 + \angle 3 + \angle 4$  is:

- A)  $327.\overline{27}^\circ$  B)  $345^\circ$  C)  $359.999^\circ$  D)  $360^\circ$  E) None of the above
2. Given the following series  $1 + 3 + 6 + 10 + 15 + 21 + \dots$ , the sum of one hundred terms is:  
A) 5050 B) 343400 C) 338350 D) 171700 E) None of the above
3. What is the million's digit of  $(1001)^{1997}$   
A) 1 B) 7 C) 9 D) 5 E) None of the above
4. The exact value of  $\sqrt{53 - 8\sqrt{15}}$  is:  
A) 4.692135253 B)  $2\sqrt{3} - 2\sqrt{5}$  C)  $4\sqrt{3} - \sqrt{5}$  D)  $-4\sqrt{3} + 5$  E) None of the above
5. The product of all prime factors of  $3^{18} - 2^{18}$  is:  
A) 77431669 B) 2800733 C) 387158345 D) 363990567 E) None of the above
6. The sum of the first  $4k$  positive integers is 2958 more than the sum of the first  $3k$  positive integers;  $k$  is:  
A) 87 B) 116 C) 31 D) 29 E) None of the above

7. Given  $BC = 50$ ,  $BD = 50\sqrt{2}$ ,  $BE = 50\sqrt{5}$ ,  $BA = 50\sqrt{10}$ ,  $\angle C$  is a right angle. What is the approximate measure of  $\angle ABD$ ?



- A)  $30^\circ$  B)  $\approx 36.869898^\circ$  C)  $\approx 26.565051^\circ$  D)  $38^\circ$  E) None of the above
8. A polynomial equation with roots 1, -2, 3 is:
- A)  $x^3 + 2x^2 - 5x + 6 = 0$  B)  $x^3 + 2x^2 - 5x - 6 = 0$  C)  $x^3 - 2x^2 - 5x + 6 = 0$   
 D)  $x^3 + 2x^2 + 5x + 6 = 0$  E) None of the above
9. From the figure below,  $FG = 5$ . Find  $HK$  in terms of  $r$ .

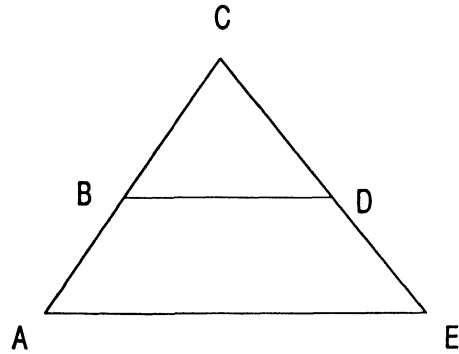


- A)  $5\sin^2 r$  B)  $5\cos^2 r$  C)  $10\sin^2 r$  D)  $5\sin r$  E) None of the above
10. What is the approximate size of the angle between the lines with equations  $2x + 5y = 3$  and  $x - 5y = 3$ ?
- A)  $26^\circ$  B)  $34^\circ$  C)  $33^\circ$  D)  $37^\circ$  E)  $42^\circ$
11. How many digits does the number  $1997^{1997}$  have?
- A)  $1997^2$  B) 6590 C) 6591 D) 6589 E) 15177

12. Three balls of radius 1 unit each are tangent to each other are placed on a plane while a fourth ball is placed on top, tangent to the other three balls. How far is the bottom of the top ball from the plane ?

A)  $\frac{3+2\sqrt{6}}{3}$  B)  $\frac{2\sqrt{6}}{3}$  C)  $\frac{2\sqrt{6}-3}{3}$  D)  $\frac{\sqrt{6}}{3}$  E) None of the above

13. ABDE is a trapezoid with  $BD = 8$  units,  $AE = 12$  units,  $AB = 6$  units, and  $\angle A = 30$  degrees. Find the length of the altitude of  $\triangle ACE$  from C:



A)  $8\sqrt{3}$  B)  $9\sqrt{3}$  C) 9 D) 12 E)  $12\sqrt{3}$

14. Find  $\sqrt{10-4i\sqrt{6}}$

A)  $-2\sqrt{3}+i\sqrt{2}$  B)  $2\sqrt{3}-i\sqrt{2}$  C)  $\pm(-2\sqrt{3}+i\sqrt{2})$  D)  $1+i\sqrt{3}$   
E) None of the above

15. A regular pyramid with a square base of side 10 units, and its height is 12 units. The radius of the inscribed sphere is:

A)  $\frac{13}{3}$  B) 4 C)  $\frac{10}{3}$  D)  $\frac{60}{13}$  E) None of the above

16. Given that  $\ln(y^2-1)-\ln(y+1)=\sin x$ , y is equal to:

A)  $e^{\sin x}-1$  B)  $1+e^{\sin x}$  C)  $\ln(x)-1$  D)  $\ln(\sin x)+1$   
E) None of the above

17. If  $\log_2 3 = a$  and  $\log_9 7 = b$ , find  $\log_2\left(\frac{7}{2}\right)$  in terms of a and b.

A)  $a^2b-1$  B)  $2ab-1$  C)  $1+2ab$  D)  $2ab$  E) None of the above

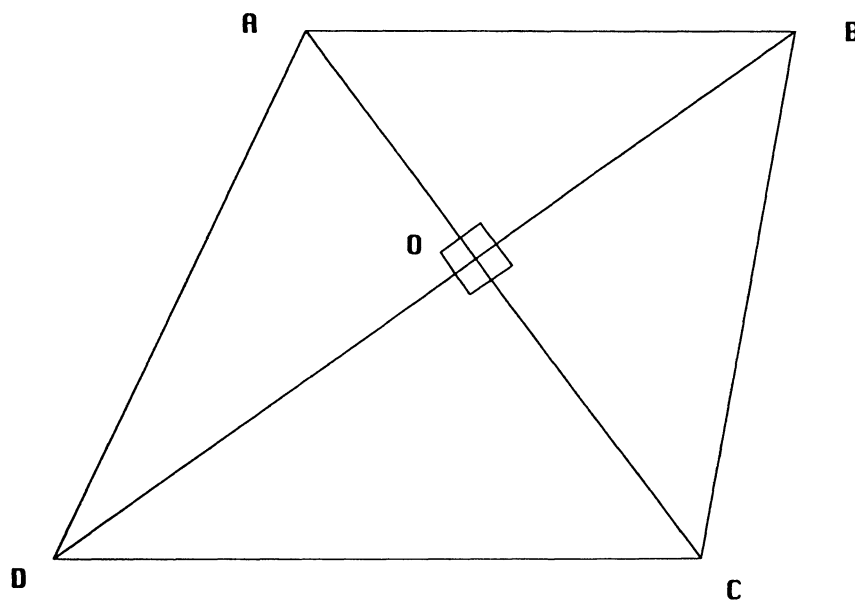
18.  $7x^2 + 6xy + 15y^2 = 144$  represents a:

- A) Circle    B) Parabola    C) Hyperbola    D) Ellipse    E) Two parallel lines

19. How many ways can five persons sit on a row of 20 chairs, if none will sit next to any other ?

- A)  $5! (20)!$     B)  $15!$     C) 524160    D) 1860480    E) None of the above

20. Given : Trapezoid ABCD with its diagonals perpendicular to each other and  $AC = 7$  units and  $BD = 9$  units. The length of its median is:



- A) 8    B)  $\frac{63}{2}$     C)  $\frac{\sqrt{130}}{2}$     D)  $\sqrt{130}$     E) None of the above

21. If we have a 5 by 5 grid. In how many ways can you select 3 squares so that no two selected squares be in the same row or column ?

- A) 4258    B) 3600    C) 600    D) 552    E) None of the above

22. What is the exact length of the longest diagonal of a regular 12-sided figure whose sides are each 10 units long?

- A)  $10\sqrt{3} + 10$     B)  $20\sqrt{3} + 20$     C)  $30 + 10\sqrt{3}$     D) 38.637033  
E)  $10(\sqrt{6} + \sqrt{2})$

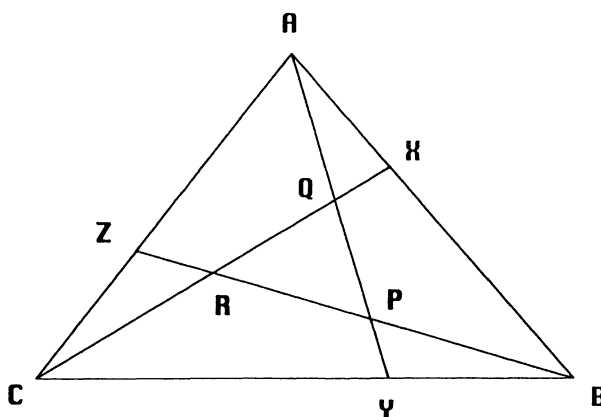
23. If the set  $\{\pm 1, \pm 2\}$  are the roots of a fourth order polynomial  $f(x)$  with integer coefficients with no common factor, and  $p$  is a prime greater than 5, what is the largest integer that divides  $f(p)$  for all allowed values of  $p$ ?

- A) 8    B) 120    C) 360    D) 600    E) 1080

24. What is the maximum number of regions that a sphere can be divided into by 7 planes?

- A) 14    B) 22    C) 42    D) 64    E) None of the above

25.  $ABC$  is an equilateral  $\Delta$  whose side is 14 units. Points  $X$ ,  $Y$ , and  $Z$  are points on the sides such that  $AX = \frac{1}{3}AB$ ,  $BY = \frac{1}{3}BC$ , and  $CZ = \frac{1}{3}AC$ .  $CX$ ,  $BZ$ , and  $AY$  are drawn to form  $\Delta PQR$ . The area of  $\Delta PQR$  is:



- A)  $\frac{49\sqrt{3}}{9}$     B)  $\frac{98\sqrt{3}}{9}$     C)  $14\sqrt{3}$     D)  $7\sqrt{3}$     E) None of the above

26. What is the average of all the distinct nine-digit numbers each of which contains only the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, each occurring exactly once.

- A) 499,999,995    B) 500,000,000    C) 555,555,555    D) 444,444,444  
E) None of the above

27. The area, in square units, of triangle  $ABC$  whose vertices are  $A=(1.2, 2.1)$ ,  $B=(3.1, 5.2)$ , and  $C=(6.1, 1.2)$  is:

- A) 16.9    B) 15    C) 8.45    D) 9.2    E) None of the above

28. If  $0^\circ \leq \theta < 360^\circ$ , The sum of the solutions of the trig equation  $2\sin(2\theta)\cos(3\theta) + \cos(3\theta) = 0$  is:

- A) 1680    B) 1830    C) 2190    D) 1800    E) None of the above

29. Find the sum of the first one hundred terms of the following series:

$$3 + 8 + 15 + 24 + 35 + \dots$$

- A) 338350    B) 171700    C) 348450    D) 25502500  
E) None of the above

30. Solve for x:  $8^{\log_8 3} - e^{\ln 5} = x^2 - 7^{\log_7 3x}$

- A) -1 and 2    B) 2 and 8    C) 3 and 7    D) 1 and 2  
E) None of the above