THE NEW ENGLAND ASSOCIATIONS OF MATHEMATICS LEAGUES

NEAML

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35th *ANNUAL MATH COMPETITION*

April 27, 2007

CANTON HIGH SCHOOL

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NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2007

Round 1: Arithmetic and Number Theory

1. ____________________

2. ____________________

3. ____________________

1. Determine the value of c such that $24_5 + 62_7 = 3c2_4$.

2. Let S be the set of primes less than 50 and let p_1 and p_2 be members of S. Determine the number of sets $\{p_1, p_2\}$ such that $(p_1 + p_2) \in S$.

3. Find the three-digit base 10 number ABC with distinct A, B, and C, and with $A \geq 1$, such that the product of the two-digit numbers AB and BC exceeds ABC by the greatest amount.

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Round 2: Algebra 1

1. ______________________

2. ______________________

3. ______________________

1. 6 less than the square of a prime number is 8 more than 5 times the number. What is the number?

2. Find the ordered pair of real numbers (x, y) satisfying the system:

$$ax + \frac{y}{b} = a$$
$$bx - \frac{y}{a} = b$$

3. If the only real solution to $x + \frac{9}{x} + y + \frac{25}{y} = 4$ is the ordered pair (a, b), determine the numerical value of $4a + 7b$.

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Round 3: Geometry

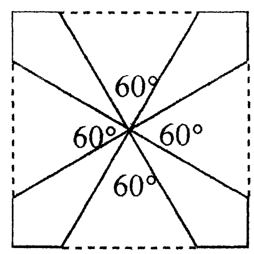
1. ____________________

2. ____________________

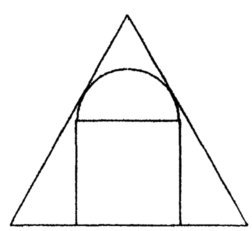
3. ____________________

1. Each side of a square has the same length as the diagonal of a 5" x 12" rectangle. A circle is inscribed in the square. Exactly how many inches are in the circumference of the circle?

2. Starting with a square, a new figure is formed by removing four equilateral triangles. Find the ratio of the perimeter of the new figure to the perimeter of the original square.



3. A figure whose base is a square and whose top is a semicircle is inscribed in an equilateral triangle of side 12 cm as shown in the diagram at the right. Determine the number of cm in the length of the radius of the semi-circle.



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Round 4: Algebra 2

1. ____________________

2. ____________________

3. ____________________

1. What is the sum of the positive integers which satisfy $\frac{(x+2)(x-5)}{x-3} \leq 0$?

2. Find the range of $y = \log_2\left(4^{x^2} + 4^{(x-1)^2}\right)$

3. The difference between the sum of the first k terms of $1^3 + 2^3 + 3^3 + \cdots + n^3$ and the sum of the first k terms of $1 + 2 + 3 + \cdots + n$ is 1980. Find k.

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Round 5: Analytic Geometry

1. ________________________

2. ________________________

3. ________________________

1. Point P is $\frac{2}{3}$ of the way from $F(-5,3)$ to $G(4,15)$. Line L is perpendicular to $\overline{FG}$ and passes through P. The equation of line L is $ax+by=c$, where a, b, and c are relatively prime integers and $a>0$. Determine the ordered triple (a, b, c).

2. Let f be a linear function whose x-intercept is positive. If, for all x, $f(x)+f^{-1}(x)=\frac{17}{4}x+2$, find the y-intercept of f.

3. Let V be the vertex of $y=k-x^2$ and let A lie on the parabola so that $AV=\sqrt{30}$. The parabola is rotated around the y-axis. Determine the area of the circle that A sweeps out.

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Round 6: Trig and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

1. If $\sin A = \dfrac{3}{7}$ and $\cos A < 0$, find $\tan 2A$.

2. If r is a solution to $x^2 + x + 1 = 0$, then determine the value of
$$\left(r+\frac{1}{r}\right)^1+\left(r^2+\frac{1}{r^2}\right)^3+\left(r^4+\frac{1}{r^4}\right)^5+\left(r^8+\frac{1}{r^8}\right)^7+\cdots+\left(r^{1024}+\frac{1}{r^{1024}}\right)^{21}.$$

3. In ΔABC, $AB = 1$, $BC = 1$, and $AC = \dfrac{1}{\sqrt{2}}$. In ΔMNP, $MN = 1$, $NP = 1$, and $m\angle MNP = 2m\angle ABC$. Find MP.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2007

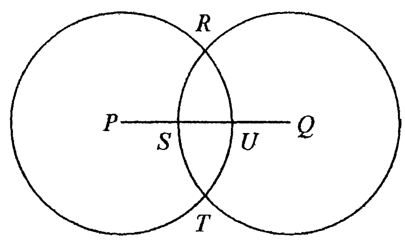
Team Round

1. ____________________ **4.** ____________________

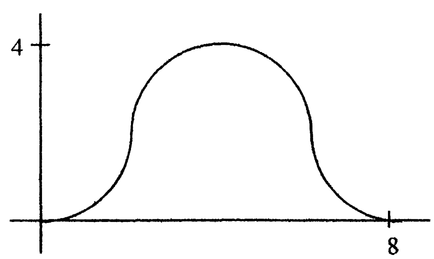
2. ____________________ **5.** ____________________

3. ____________________ **6.** ____________________

1. Two circles of radius 2 intersect the segment $\overline{PQ}$ connecting their centers at trisection points of the segment. The area of the region bounded by $\overline{SU}$, arcRS, and arcRU equals $A\cos^{-1}B - C$ for real numbers A, B, and C with $B = m\angle RPU$. Compute the product $A \cdot B \cdot C$.



2. The function f is defined on the set of real numbers. One period of the graph of f is shown. It consists of four quarter-circles of radius 2. Determine the value of $f\left(\frac{2007}{2}\right)$.



3. Determine the lattice point (x, y, z) with the largest value of z, such that (x, y, z) satisfies both $7x + 6y + 2z = 272$ and $x - y + z = 16$ and that lies in the first octant where $x, y, z > 0$.

4. Find four sets of three positive real numbers, such that given any two of the three numbers, one of the two is either twice the other or the square of the other.

5. Given $A(-4,0)$ and $B(4,0)$, the area enclosed by the set of points P such that $(PA)^2 + (PB)^2 = k$ is 9. Determine the value of k.

6. Points A and B lie on the line $y = -\frac{x}{2} + 4$, point $P = (6, 6)$ and $m\angle APB = 45°$. Determine the smallest possible area for triangle APB.

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Answer Sheet

Round 1

1. 2
2. 6
3. 897

Round 2

1. 7
2. $(1,0)$
3. 23

Round 3

1. 13π
2. $\dfrac{3+\sqrt{3}}{3}$
3. $\dfrac{3\sqrt{3}}{2}$

Round 4

1. 9
2. $y \geq \dfrac{3}{2}$
3. 9

Round 5

1. $(3,4,47)$
2. $-\dfrac{2}{3}$
3. 5π

Round 6

1. $-\dfrac{12\sqrt{10}}{31}$
2. -11
3. $\dfrac{\sqrt{7}}{2}$

Team

1. $\dfrac{9\sqrt{7}}{4}$
2. $\dfrac{4+\sqrt{15}}{2}$
3. $(8,25,33)$
4. $\{1,1,1\},\{1,2,1\},\{4,8,16\},$ $\{2^{1/3},2^{2/3},2^{4/3}\},\{2^{-1/3},2^{-2/3},2^{-4/3}\}$
5. $\dfrac{18+32\pi}{\pi}$
6. $20\sqrt{2}-20$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2007 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. In base 10, $14 + 44 = 58$. $58 = 322_4$

2. If the sum of two primes is prime, then one of the primes must be 2. We have $2 + 3 = 5$, $2 + 5 = 7$, $2 + 11 = 13$, $2 + 17 = 19$, $2 + 29 = 31$, and $2 + 41 = 43$. This makes 6 pairs.

3. $(10A+B)(10B+C)-(100A+10B+C)=100AB-100A+10B^2-10B+10AC+BC-C$ $= 100A(B-1)+10B(B-1)+C(B-1)+10AC = (100A+10B+C)(B-1)+10AC$. Now, if $A = 9$, $B = 8$, and $C = 7$ we have $987\cdot 7+10\cdot 9\cdot 7 = 7539$. If $A = 8$, $B = 9$, and $C = 7$, we have $897\cdot 8+10\cdot 8\cdot 7 = 7736$. Hence, $ABC = 897$.

Round 2 Algebra 1

1. $a^2-6=5a+8$, $a^2-5a-14=0$. The prime number solution is 7.

2. Multiply the top by b and the bottom by a and obtain

$$\begin{aligned} abx + y &= ab \\ abx - y &= ab \end{aligned}$$

Subtracting gives $y = 0$ and $x = 1$.

Ans: $(1, 0)$.

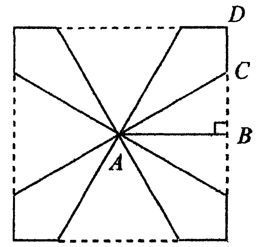
3. $x+\dfrac{9}{x}+y+\dfrac{25}{y}=4 \rightarrow \dfrac{x^2+9}{x}+\dfrac{y^2+25}{y}=4 \rightarrow$

$\dfrac{x^2+9}{x}+\dfrac{6x}{x}+\dfrac{y^2+25}{y}-\dfrac{10x}{x}-6+10=4 \rightarrow \dfrac{(x+3)^2}{x}+\dfrac{(y-5)^2}{y}=0$. The only solution is $(-3, 5)$, so $4(-3)+7(5) = 23$.

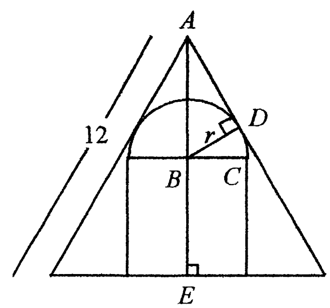
Round 3 – Geometry

1. The square is 13" on a side which is also the diameter of the circle. Ans. 13π

2. Let $BC = x$. Since ΔABC is a 30-60-90 right triangle, then $AB = x\sqrt{3}$ and $AC = 2x$. Thus, the side of the square is $2x\sqrt{3}$, making the perimeter equal to $8x\sqrt{3}$. The sum of AC and CD equals $2x + \left(x\sqrt{3} - x\right) = x + x\sqrt{3}$. Eight such sums make up the perimeter of the figure so its perimeter is $8x\sqrt{3} + 8x$. The ratio of perimeters equals $\dfrac{8x\sqrt{3} + 8x}{8x\sqrt{3}} = \dfrac{3 + \sqrt{3}}{3}$.



3. Draw the radius $\overline{BD}$ perpendicular to the side of the equilateral triangle. Since ΔABD is a 30-60-90 triangle, $AB = 2r$, and since $BC = r$, then the side of the square is $2r$. Thus, $AE = 4r = 6\sqrt{3} \to r = \dfrac{3\sqrt{3}}{2}$.



Round 4 – Algebra 2

1. The real solutions are $x \le -2$ or $-3 < x \le 5$. The positive integers and 4 and 5.

2. By the Arithmetic-Geometric Mean Inequality, $4^{x^2} + 4^{(x-1)^2} \ge 2\sqrt{4^{x^2} \cdot 4^{(x-1)^2}} = 2\sqrt{4^{2x^2-2x+1}}$. The minimum value of $2x^2 - 2x + 1$ occurs at $x = -\dfrac{-2}{4} = \dfrac{1}{2}$ and the min equals $2\left(\dfrac{1}{2}\right)^2 - 2\left(\dfrac{1}{2}\right) + 1 = \dfrac{1}{2}$. Hence, $4^{x^2} + 4^{(x-1)^2} \ge 2\sqrt{4^{1/2}} = 2^{3/2}$. The minimum of $\log_2\left(4^{x^2} + 4^{(x-1)^2}\right)$ is $\log_2 2^{3/2} = 3/2$. Range: $y \ge \dfrac{3}{2}$.

3. The equation $\left(\frac{k(k+1)}{2}\right)^2 - \frac{k(k+1)}{2} = 1980$ is quadratic in $\frac{k(k+1)}{2}$. Let $x = \frac{k(k+1)}{2}$ giving $x^2 - x - 1980 = 0 \to (x-45)(x+44) = 0$. Thus, $x = 45$, making $\frac{k(k+1)}{2} = 45$ and so $k = \boxed{9}$.

Round 5 – Analytic Geometry

1. $m\overline{FG} = \frac{4}{3}, m_L = -\frac{3}{4}$, $P(1,11)$, $y - 11 = -\frac{3}{4}(x-1)$

2 Let $f(x) = mx + b$, then $f^{-1}(x) = \frac{x}{m} - \frac{b}{m}$ making $m + \frac{1}{m} = \frac{17}{4} \to 4m^2 - 17m + 4 = 0$ $\to (4m-1)(m-4) = 0 \to m = \frac{1}{4}$ or 4. Now $b - \frac{b}{m} = 2$ so if $m = \frac{1}{4}$ then $b - 4b = 2$ $\to b = -\frac{2}{3}$. Since the equation $f(x) = \frac{x}{4} - \frac{2}{3}$ has a positive x-intercept, it satisfies the problem and the answer is $-\frac{2}{3}$. For $m = 4$, the equation is $f(x) = 4x + \frac{8}{3}$, but its x-intercept is negative.

3. Given $V(0,k)$ and $A(x, k - x^2)$, then $AV = \sqrt{x^2 + \left(k - x^2 - k\right)^2}$; $x^4 + x^2 = 30$. Then $\left(x^2 + 6\right)\left(x^2 - 5\right) = 0 \to x^2 = 5$. Since x is the radius, the area is 5π.

Round 6 – Trig and Complex Numbers

1. $\cos A = \frac{-2\sqrt{10}}{7}$, $\tan A = -\frac{3}{2\sqrt{10}} = -\frac{3\sqrt{10}}{20}$, $\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = -\frac{12\sqrt{10}}{31}$

2. $x^2 + x + 1 = 0 \to x^2 + 1 = -x \to x + \frac{1}{x} = -1$. Thus, $\left(x + \frac{1}{x}\right)^1 = -1$. Also, $\left(x + \frac{1}{x}\right)^2 = (-1)^2 \to x^2 + 2 + \frac{1}{x^2} = 1 \to x^2 + \frac{1}{x^2} = -1$. Thus, $\left(x^2 + \frac{1}{x^2}\right)^3 = -1$. If we obtain $x^4 + \frac{1}{x^4}$ in a similar fashion by squaring $x^2 + \frac{1}{x^2}$ we will again obtain -1. Thus, each expression represents -1 to an odd power so the sum equals $\boxed{-11}$.

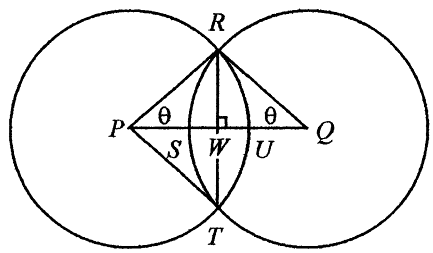
3. Using ΔABC with $m\angle ABC = \theta$, we have $\left(\frac{1}{\sqrt{2}}\right)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1\cos\theta \rightarrow \cos\theta = \frac{3}{4}$.

Since $\cos 2\theta = 2\cos^2\theta - 1$, then $\cos\angle MNP = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$. Then

$MP^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \frac{1}{8} \rightarrow MP = \frac{\sqrt{7}}{2}$.

Team Round

1. Since $PR = RQ = 2$, then $SU = 1$, making $SW = 1/2 \rightarrow \cos\theta = 3/4$. Thus, the area of the sector RPU is $\frac{\cos^{-1}(3/4)}{2\pi} \cdot \pi \cdot 2^2 = 2\cos^{-1}(3/4)$. Since $RW = \sqrt{7}/2$, the area of



$\Delta RPW = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{\sqrt{7}}{2}$. Thus, the area of the region bounded by $\overline{RW}$, $\overline{WU}$, and $\overset{\frown}{RU}$ is $2\cos^{-1}\frac{3}{4} - \frac{3\sqrt{7}}{8}$. The area of the desired region is $4\cos^{-1}\frac{3}{4} - \frac{3\sqrt{7}}{4}$. The desired product is $4 \cdot \frac{3}{4} \cdot \frac{3\sqrt{7}}{4} = \frac{9\sqrt{7}}{4}$.

2. The period is 8 and since $\frac{2007}{2} \div 8 = 125 + \frac{7}{16}$, $f\left(\frac{2007}{2}\right) = f\left(8 \cdot \frac{7}{16}\right) = f\left(\frac{7}{2}\right)$. To obtain the y-value, determine the value of y when $x = 3.5$ using the equation $(x-4)^2 + (y-2)^2 = 4$. Thus, $(3.5-4)^2 + (y-2)^2 = 4 \rightarrow y = \frac{4+\sqrt{15}}{2}$.

3. $7x + 6y + 2z - 2(x - y + z) = 272 - 2 \cdot 16 \rightarrow 5x + 8y = 240 \rightarrow x = 48 - \frac{8}{5}y$. Let $y = 5k$ for k an integer, then $x = 48 - 8k$, and substituting for both x and y into $x - y + z = 16$ we obtain $z = 13k - 32$. For x to be positive $k \le 5$ and for z to be positive $k \ge 3$. Thus, there are three lattice points of intersection lying in the first octant. For $k = 3$ we have (24, 15, 7), for $k = 4$ we have (16, 20, 20), and for $k = 5$ we have (8, 25, 33).

4. First note that {1, 1, 1} satisfies the conditions. Next, given $\{a, 2a, a^2\}$, the first two and the first and third satisfy the conditions. It remains to consider the second and third. There are 4 cases:

i) $a^2 = 2(2a) \to a = 0, 4$. 0 fails so we have $a = 4$ giving {4, 8, 16}.

ii) $a^2 = (2a)^2 \to a = 0$

iii) $2a = (a^2)^2 \to a^3 = 2 \to a = 2^{1/3} \to \{2^{1/3}, 2^{4/3}, 2^{2/3}\}$

iv) $2a = 2(a^2) \to a = a^2 \to a = 0, 1$. From $a = 1$ we have {1, 2, 1}.

Had we considered $\{a, a^2, a^4\}$ and set $a = 2a^4 \to a^3 = 2^{-1}$, making $a = 2^{-1/3}$, giving the fifth solution $\{2^{-1/3}, 2^{-2/3}, 2^{-4/3}\}$.

5. $(x+4)^2 + y^2 + (x-4)^2 + y^2 = k \to 2x^2 + 2y^2 = k - 32$. Thus, $x^2 + y^2 = \frac{k-32}{2}$. If the circle has area of 9, then its radius is $\frac{3}{\sqrt{\pi}}$ so $\frac{k-32}{2} = \frac{9}{\pi} \to k = \frac{18}{\pi} + 32 = \frac{18+32\pi}{\pi}$.

6. Given $y = -\frac{x}{2} + 4 \leftrightarrow x + 2y - 8 = 0$, the distance from P to the line is $\frac{|6 + 2\cdot 6 - 8|}{\sqrt{1^2+2^2}} = 2\sqrt{5}$. The minimum area will occur when the base $\overline{AB}$ is the least and that occurs when APB is an isosceles triangle. Then the altitude from P bisects $\angle APB$ and if x is half the base, then

$\tan(22.5) = \frac{x}{2\sqrt{5}}$. Since $\tan 45° = 1 = \frac{2\tan(22.5)}{1-\tan^2(22.5)}$, then $1 = \frac{\frac{x}{\sqrt{5}}}{1-\frac{x^2}{20}}$ giving

$x^2 + 4\sqrt{5} - 20 = 0 \to x = 2\sqrt{10} - 2\sqrt{5}$. The area is $x\left(2\sqrt{5}\right) = \boxed{20\sqrt{2} - 20}$.