THE NEW ENGLAND ASSOCIATIONS OF MATHEMATICS LEAGUES

NEAML

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34th *ANNUAL MATH COMPETITION*

April 28, 2006

CANTON HIGH SCHOOL

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NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2006

Round 1: Arithmetic and Number Theory

1. ______________________

2. ______________________

3. ______________________

1. Write in simplest form: $\left(\dfrac{63^2 - 36^2}{297}\right)^{-\frac{3}{2}}$.

2 Find the value of C given $2.3_4 + C_2 = 21.2_8$ where 2, 4, and 8 are base indexes.

3. Ed is on an 8-day cycle in which he works 7 consecutive days, then gets 1 day off. Sue is on a 6-day cycle in which she works 4 consecutive days, then gets 2 days off. If they begin working on Jan. 1, how many days will they have off together in a full year of 365 days?

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Round 2: Algebra 1

1. ______________________

2. $a =$ $b=$ $c=$ $d=$ ______

3. ______________________

1. How many multiples of 3 satisfy the inequality $2 \le \left|\frac{3m+1}{4}\right| < 7$?

2. The domain of $y = f(x)$ is $-1 \le x \le 4$ and the range is $1 \le y \le 10$. Let $g(x) = -3f(x-2)+4$. If the domain of $g(x)$ is $a \le x \le b$ and the range of $g(x)$ is $c \le y \le d$, determine the values of a, b, c, and d.

3. Let w be the wholesale price of a carpet per square foot and r be the retail price of a carpet per square foot. Note that w and r are integers with $r > w$. If w drops by \$1 and r drops by \$1, but the ratio of r to w goes up by 10%, find the least possible value of r.

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Round 3: Geometry

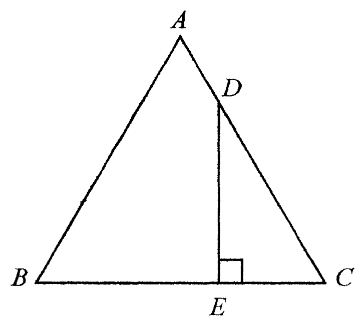
1. ______________________

2. ______________________

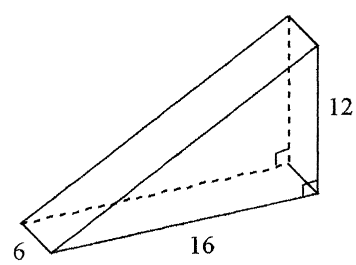
3. ______________________

1. $ABCD$ has perpendicular diagonals. If $AB = 1$ cm, $BC = 4$ cm, and $CD = 8$ cm, determine the number of centimeters in AD.

2. ABC is an equilateral triangle of side 4 cm. If the ratio of the area of DEC to the area of $ADEB$ is 1: 3, find the number of centimeters in the length of $\overline{DE}$.



3. I shall seize a city inimical to Marduk. My uncompleted siege ramp has width 6 m, length 16 m and height 12 m. When completed, keeping the width the same, and the length and height in the same ratio, its volume will be 625 m^3. When it is completed, what will the number of meters in its length be?



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Round 4: Algebra 2

1. ______________________

2. ______________________

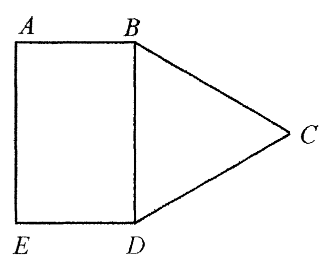
3. ______________________

1. The radius and height of a cylinder are integers. If the total surface area is numerically equal to the volume, determine the largest possible number of units in the height of the cylinder.

2. Let x be an integer and let $[x]$ be the greatest integer function. If $[\log(\log x)] = 3$, compute the maximum number of digits that x could have.

3. $ABDE$ is a rectangle and BCD is an equilateral triangle. The perimeter of $ABCDE$ is 33 in

   Find the number of inches in the length of $\overline{AE}$ that maximizes the area of $ABCDE$



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Round 5: Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. Point A lies on $y = 2x$ and the sum of its coordinates is 12. Point B lies on the x-axis, and $\overline{AB}$ is perpendicular to $y = 2x$. Let O be the origin. Determine the number of square units in the area of ΔAOB.

2. A rhombus with a side of 4 units and a $60°$ angle is inscribed in an ellipse. Determine the number of units in the distance between the focal points of the ellipse if the diagonals of the rhombus are the axes of the ellipse.

3. For positive integers x and k, let the slope of the line connecting $(1,1)$ and $\left(x, x^3\right)$ be k.

   Determine the number of values of k less than 100.

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Round 6: Trig and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

1. For real a and b, if the solutions to $x^{2005} = 1$ are written in the form $a + bi$, determine the number of distinct ordered pairs (a, b) such that both a and b are positive.

2. In quadrilateral $ABCD$, if $\cos A + \cos C = 0$ and $\angle B$ is acute, determine the value of $\dfrac{\tan B}{\tan D}$.

3. For $S_{2n} = i + 2i^2 + 3i^3 + \cdots + ni^n + \dfrac{n+1}{i^1} + \dfrac{n+2}{i^2} + \cdots + \dfrac{2n}{i^n}$, determine the number of values of n where $1 \le n \le 2005$ for which S_{2n} is a positive real number

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2006

Team Round

1. ____________________ **4.** ____________________

2. ____________________ **5.** ____________________

3. ____________________ **6.** ____________________

1. A cylindrical water tank is lying on its side. Its radius is 4 ft and its length is 9 ft. It is filled with water to a depth of 6 feet. If the tank is turned upright and set on its base, determine the number of feet in the height of the water in the cylinder.

2. For $f(x) = mx$, the area of the triangle formed by (0, 0), the first quadrant point $(a, f(a))$ and the reflection of that point across $y = x$ is 1000. If m and a are positive integers, determine the value of m.

3. Determine the number of positive integers less than 100 for which the number of positive divisors is prime.

4 If $\dfrac{2a+b}{a+2b+c} = \dfrac{1}{2}$ and $\dfrac{c-b}{c-b+a} = \dfrac{1}{3}$, determine the ratio $a : b : c$ where a, b, and c are positive and their greatest common factor is 1.

5. The solution in $\left[0, \dfrac{\pi}{2}\right)$ to $\dfrac{\cos(4x) - 1}{\cos^2 x} = -7$ can be written as $\tan^{-1}(b)$. Find b in simplest form.

6. How many solutions are there to the equation $\left|\left(\left|\left(\left|x\right| - 1\right)\right| - 2\right| \right) - 3\right| = 2$?

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Answer Sheet

Round 1

1. $\frac{1}{27}$
2. 1110.1
3. 15

Round 2

1. 5
2. 1, 6, -26, 1
3. 12

Round 3

1. 7
2. $\sqrt{6}$
3. $\frac{50}{3}$

Round 4

1. 6
2. 10000
3. $6+\sqrt{3}$

Round 5

1. 80
2. $4\sqrt{2}$
3. 8

Round 6

1. 501
2. -1
3. 501

Team

1. $6+\frac{9\sqrt{3}}{4\pi}$
2. 9
3. 32
4. 2 : 5 : 6
5. $\sqrt{7}$
6. 7

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2006 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. Treating the numerator as a difference of 2 squares and the denominator as $11 \cdot 27$, get $\dfrac{1}{27}$

2. $2.3_4 = 2.75_{10}$, $21.2_8 = 17.25_{10}$, Subtract and change to base 2.

3. Ed's free day is always a multiple of 8 and Sue's free day is either a multiple of 6 or one less than a multiple of 6. The latter is always odd and so they aren't free together on such a day. Thus, $8k = 6n \rightarrow n = \dfrac{4}{3}k$. Since k must be a multiple of 3, set $k = 3m$ giving free days together on days whose number is $24m$. Divide 365 by 24 and obtain $\boxed{15}$ plus a remainder.

Round 2 Algebra 1

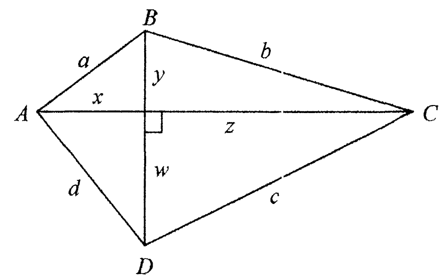
1. An equivalent inequality is $8 \le 3m + 1 < 28$.

2. Domain: $-1 \le x - 2 \le 4 \rightarrow \boxed{1 \le x \le 6}$. Range: if the range of $y = f(x)$ is $1 \le y \le 10$, then the range of $y = -3f(x-2)$ is $-30 \le y \le -3$, making the range of $y = -3f(x-2) + 4$ the set of y-values such that $\boxed{-26 \le y \le 1}$.

3. $\dfrac{r-1}{w-1} = \dfrac{11}{10} \cdot \dfrac{r}{w} \rightarrow 10w(r-1) = 11r(w-1) \rightarrow r = \dfrac{10w}{11-w}$. Let $z = 11 - w$, giving $r = \dfrac{10(11-z)}{z} = \dfrac{110}{z} - 10$. This substitution allows us to determine the values of r more easily. For r to be positive, z must be a divisor of 110 smaller than 11. If $z = 10$, then $r = 1$, but w also equals 1. If $z = 5$, then $r = 12$ and $w = 6$. Hence, the least value of r is $\boxed{12}$.

Round 3 – Geometry

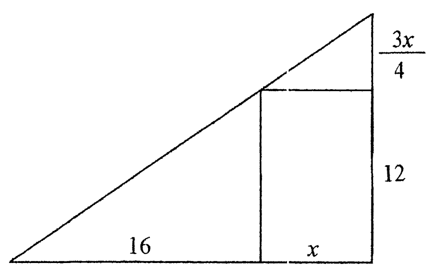
1. From the Pythagorean Theorem we can deduce that $a^2 + c^2 = b^2 + d^2$. Hence, $1^2 + 8^2 = 4^2 + AD^2 \rightarrow AD = \boxed{7}$.



2. Let $EC = x$, $DE = x\sqrt{3} \rightarrow \dfrac{a\Delta DEC}{a\Delta ABC} = \dfrac{\frac{x}{2}\left(x\sqrt{3}\right)}{\frac{4^2\sqrt{3}}{4}} = \dfrac{x^2}{8} = \dfrac{1}{4}$. Thus, $x = \sqrt{2}$, making

$DE = \sqrt{2} \cdot \sqrt{3} = \boxed{\sqrt{6}}$.

3. Convert to 2-space by dividing the volume by 6. Thus, $\frac{1}{2}(16 + x)\left(12 + \frac{3x}{4}\right) = 104\frac{1}{6} \rightarrow 9x^2 + 288x - 196 = 0$ $\rightarrow (3x - 2)(3x + 98) = 0 \rightarrow x = \frac{2}{3}$. The complete length is $\boxed{16\frac{2}{3} = \frac{50}{3}}$. Note: this problem is based on an Old Babylonian siege ramp problem BM 85194 circa 1600 BC.



Round 4 – Algebra 2

1. $2\pi r^2 + 2\pi rh = \pi r^2 h \rightarrow 2r = rh - 2h \rightarrow h = \dfrac{2r}{r-2} = 2 + \dfrac{4}{r-2}$ by division. Clearly, h is maximized when $r = 3$, making $h = \boxed{6}$.

2. $\lfloor \log(\log x) \rfloor = 3 \rightarrow 3 \le \log(\log x) < 4 \rightarrow 10^3 \le \log x < 10^4 \rightarrow 10^{1000} \le x < 10^{10,000}$. If x could equal $10^{10,000}$ it would have 10,001 digits. But it can't so the maximum number of digits is $\boxed{10,000}$.

3. Let $AE = x$ and $AB = y$. Then $3x + 2y = 33 \rightarrow y = \frac{33}{2} - \frac{3}{2}x$ and the area of the figure is $f(x) = x\left(\frac{33}{2} - \frac{3}{2}x\right) + \frac{x^2\sqrt{3}}{4} = \left(\frac{\sqrt{3}-6}{4}\right)x^2 + \frac{33}{2}x$. This is concave down so the maximum occurs at the vertex, at $x = \frac{-\frac{33}{2}}{2\left(\frac{\sqrt{3}-6}{4}\right)} = \frac{33}{6-\sqrt{3}} = \frac{33}{6-\sqrt{3}} \cdot \frac{6+\sqrt{3}}{6+\sqrt{3}} = \boxed{6+\sqrt{3}}$.

Round 5 – Analytic Geometry

1. A is (4, 8), the equation of $\overline{AB}$ is x + 2y = 20, giving B(20, 0). The area of ΔAOB =80.

2 If the sides are each 4 and an acute angle is 60°, then the rhombus can be divided into four 30-60-90 right triangles with sides of 2 and $2\sqrt{3}$. The minor axis is 4 and the major axis is $4\sqrt{3}$, making the distance from the center of the ellipse to a focal point equal to $\sqrt{\left(2\sqrt{3}\right)^2 - 2^2} = 2\sqrt{2}$. Ans: $\boxed{4\sqrt{2}}$..

3. $\frac{x^3-1}{x-1} = k \rightarrow x^2 + x + 1 = k < 100$. Clearly, $x = 2, 3 \ldots, 9$, giving values of k from $k = 7$ to $k = 91$. Thus, there are $\boxed{8}$ values of k.

Round 6 – Trig and Complex Numbers

1. There are 2005 complex solutions, one of which lies at (1, 0) on the x-axis and the rest lie uniformly in the 4 quadrants where they form the vertices of a regular polygon with 2005 sides. The answer is 2004/4 = $\boxed{501}$.

2. Since $\cos A = -\cos C$ then A and C are supplementary as are B and D. Therefore, $\tan B = -\tan D$ giving $\frac{\tan B}{\tan D} = \boxed{-1}$.

3. $S_2 = 1i + \frac{2}{i} = i - 2i = -i$. $S_4 = 1i + 2i^2 + \frac{3}{i} + \frac{4}{i^2} = i - 2 - 3i - 4 = -6 - 2i$.

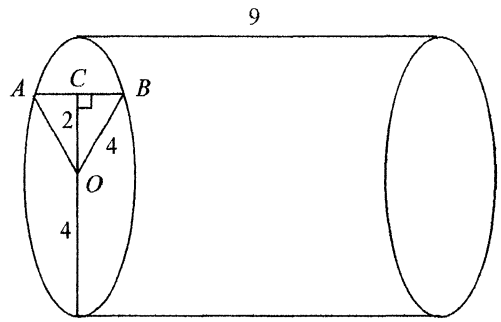
$S_6 = i + 2i^2 + 3i^3 + \frac{4}{i} + \frac{5}{i^2} + \frac{6}{i^3} = i - 2 - 3i - 4i - 5 + 6i = -7 + 0i$.

$S_8 = i + 2i^2 + 3i^3 + 4i^4 + \frac{5}{i} + \frac{6}{i^2} + \frac{7}{i^3} + \frac{8}{i^4} = i - 2 - 3i + 4 - 5i - 6 + 7i + 8 = 4 + 0i$.

If $2n$ is a multiple of 8, then S_{2n} is a positive real number. Thus, if n is a multiple of 4, then $2n$ is a multiple of 8, so there are $\left\lfloor\frac{2005}{4}\right\rfloor = [501.25] = \boxed{501}$ values of n.

Team Round

1. The water is up to the height of $\overline{AB}$. Since $OC = 2$, ΔOBC is a 30-60-90 triangle with $m\angle BOC = 60$. The volume of the water equals the height 9 times the area of the region consisting of two sectors with a 120 degree central angle plus $\Delta AOB = 9\left(2 \cdot \frac{1}{3}\pi 4^2 + \frac{1}{2} \cdot 4\sqrt{3} \cdot 2\right) = 96\pi + 36\sqrt{3}$. If the cylinder is turned upright, the volume of the water equals $\pi 4^2 \cdot h$. From $96\pi + 36\sqrt{3} = 16\pi h$, $h = \boxed{6 + \frac{9\sqrt{3}}{4\pi}}$.



2. Let $f(a) = ma$, then we have the points (0, 0), (a, ma), and (ma, a). Using determinants we have $\frac{1}{2}\begin{vmatrix} ma & a \\ a & ma \end{vmatrix} = 1000 \to a^2(m^2 - 1) = 2000$. We search factor pairs of 2000 such that one will be a perfect square and the other will be one less than a perfect square. We find that $2000 = 25 \bullet 80$ fits the bill. So $m^2 - 1 = 80$ makes $\boxed{m = 9}$.

3. If the number of divisors of N is 2 then N must be prime. Count the number of primes less than 100; there are 25. If the number of divisors of N is 3, then N must be the square of a prime, namely 4, 9, 25, or 49, giving 4 more values for N. If the number of divisors is 5, then N can only be 2^4 or 3^4. If the number of divisors is 7, then N is 2^6. The answer is $25 + 4 + 2 + 1 = \boxed{32}$.

4. $\frac{2a + b}{a + 2b + c} = \frac{1}{2} \to 4a + 2b = a + 2b + c \to 3a = c$. Also, $\frac{c - b}{c - b + a} = \frac{1}{3} \to 3c - 3b = c - b + a \to 5a = 2b \to b = \frac{5a}{2}$. Thus, $a : b : c = a : \frac{5}{2}a : 3a \to \boxed{2 : 5 : 6}$.

5. $\frac{\cos 4x - 1}{\cos^2 x} = \frac{\cos(2(2x)) - 1}{\cos^2 x} = \frac{\left(2\cos^2 2x - 1\right) - 1}{\cos^2 x} = \frac{2\left(\cos^2 2x - 1\right)}{\frac{\cos 2x + 1}{2}} =$

$\frac{4(\cos 2x - 1)(\cos 2x + 1)}{\cos 2x + 1} = 4\cos 2x - 4 = -7$. Thus, $4\cos 2x = -3$, giving

$2\cos^2 - 1 = -\frac{3}{4} \rightarrow \cos^2 x = \frac{1}{8}$, giving $\sin^2 x = \frac{7}{8} \rightarrow \tan^2 x = 7$. Thus, $\tan x = \sqrt{7}$ for

$0 \le x < \frac{\pi}{2}$ making $x = \tan^{-1}\sqrt{7}$. Thus, $b = \boxed{\sqrt{7}}$.

6. $\left|\left||x| - 1\right| - 2\right| - 3 = \pm 2 \rightarrow \left|\left||x| - 1\right| - 2\right| = 1 \text{ or } 5 \rightarrow \left||x| - 1\right| - 2 = \pm 1 \text{ or } \pm 5 \rightarrow$

$\left||x| - 1\right| = 1, -3, 3, \text{ or } 7$. Reject -3. Thus, $|x| - 1 = \pm 1, \pm 3, \text{ or } \pm 7$ Thus,

$|x| = 8, -6, 4, -2, 0$. Reject -6 and -2 $\rightarrow x = \pm 8, \pm 4, \pm 2, \text{ or } 0$. There are $\boxed{7}$ solutions.