**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

**Round 1  Arithmetic and Number Theory**

**1.** ________________________

**2.** ________________________

**3.** ________________________

1. Mary needs to get a B average in math. Her test scores so far are 72, 81, 77, and 92. What is the range of scores she can get on the final test of the term so her average will be a B for the term? The maximum score on any one test is 100. (A grade of B is given for an average without rounding between 80 and 89 inclusive.) Express your answer as the ordered pair (min, max) where min is the minimum score she can get and max is the maximum score.

2. Two fractions are such that their numerators and denominators are the four one digit prime numbers. Determine the largest possible sum for these fractions.

3. Determine the smallest integer value of $n$, $n > 1$, such that the set of consecutive counting numbers, $S = \{1, 2, 3, 4, . . . , n\}$ contains exactly 8.75 times as many multiples of 2 as of 17.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

**Round 2   Algebra 1**

**1.** ________________________

**2.** ________________________

**3.** ________________________

1. If $25^{3x-2} = \left(\frac{1}{125}\right)^{4-x}$, determine the value of $x$.

2. In Tom's pocket, there are four times as many quarters as nickels but there are not enough nickels to make change for a quarter. 15 more than the square of the number of nickels is equal to 8 times the number of nickels. If there are no other coins in Tom's pocket, what is the total value of these coins.

3. If $a \oplus b = ab + b$, then find all $a$ such that the following equation has two distinct real solutions in $x$:
   $x \oplus (a \oplus x) = -\frac{1}{8}$.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

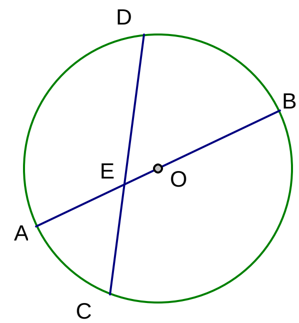
**Round 3 – Geometry**

**1.** ________________________

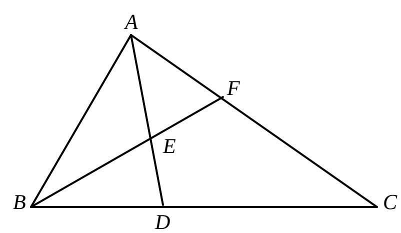
**2.** ________________________

**3.** ________________________

1. In circle $O$, diameter $\overline{AB}$ has length 12 units. Chord $\overline{CD}$ intersects $\overline{AB}$ at $E$. $AE{:}EB = 3{:}5$ and $CE = 4$. Find the number of units in the length of $\overline{ED}$.




2. In $\Delta ABC$, $\overline{BF}$ bisects $\angle ABC$ and $\dfrac{BE}{EF} = \dfrac{BC}{BA}$. If $AB = 2$ and $BC = 3$, express the ratio $\dfrac{AE}{ED}$ in the form $\dfrac{a}{b}$ where $a$ and $b$ are relatively prime.




3. In a 5-12-13 triangle, a segment is drawn parallel to the hypotenuse and 1/3 the way from the hypotenuse to the opposite vertex. Another segment is drawn parallel to the first and 1/3 the way from the previous segment to the opposite vertex. Determine the number of square units in the trapezoid whose bases are the two drawn segments.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

**Round 4 – Algebra 2**

**1.** ________________________

**2.** ________________________

**3.** ________________________

1. $A$ is the sum of all terms of an infinite geometric sequence whose fourth term is 4 and whose seventh term is $\frac{1}{2}$. $B$ is the sum of all terms of an infinite geometric sequence whose fifth term is $\frac{1}{3}$ and whose common ratio is $\frac{1}{3}$. Determine the value of $A + B$.

2. If $f(x) = 2x + 1$, determine all values of $x$ such that $f\left(\frac{1}{f^{-1}(x)}\right) = f^{-1}\left(\frac{1}{f(x)}\right)$. Note: $f^{-1}$ is the inverse function of $f$.

3. If $[x]$ stands for the greatest integer less than or equal to $x$, solve $x[x] = 2005$.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

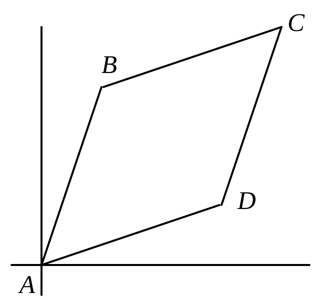
**Round 5 – Analytic Geometry**

**1.** ____________________

**2.** ( , , )

**3.** ____________________

1. Given $A(0,0)$, $ABCD$ is a rhombus of side 5 where the slope of $\overline{AB}$ is 2 and the slope of $\overline{AD}$ is $\frac{1}{2}$. Find the coordinates of $C$.




2. The equation of a hyperbola is $xy - 4y + 6x - 28 = 0$. A circle is such that its diameter is the segment connecting the vertices of the hyperbola. If the equation of the circle is written as $(x-h)^2 + (y-k)^2 = r^2$, where $r$ is its radius, determine the ordered triple $(h,k,r)$.

3. Let $k$ be a positive number. If the area of the region bounded by the $x$-axis and the graph of $y = |x| + |x-1| - k$ is 16, determine the value of $k$.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

**Round 6 – Trig and Complex Numbers**

**1.** ________________________

**2.** ________________________

**3.** ________________________

1. Simplify $\frac{(8-i)^2}{1-2i}$ where $i=\sqrt{-1}$. Express the answer in the form $a + bi$

2. If $\tan(a+b)=\frac{7}{3}$ and $\tan b=\frac{2}{3}$, and $a$ and $b$ are acute, what is $\tan(b-a)$?

3. For $0 \le x \le 2p$, if $\frac{\cos 2x+1}{\cos 2x-1}=-\sec^2 x$, determine the value of $\cos^2 x$.

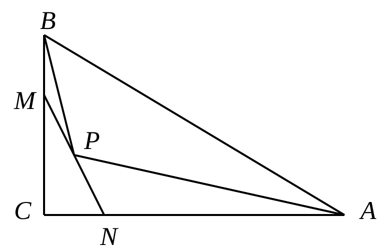
**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2005**

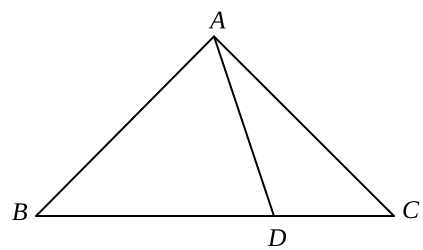
**Team Round**

| | |
|---|---|
| **1.** ________________________ | **4.** ____________________ |
| **2.** __(______,______)_______ | **5.** ____________________ |
| **3.** ________________________ | **6.** ____________________ |

1. In $\Delta ABC$, $AC = 8$, $BC = 6$, and $AB = 10$. If $m\angle MNC = m\angle ABC$ and $P$ is the midpoint of $\overline{MN}$, determine the length of $\overline{MN}$ so that the area of $\Delta BPA$ is half the area of $\Delta ABC$.



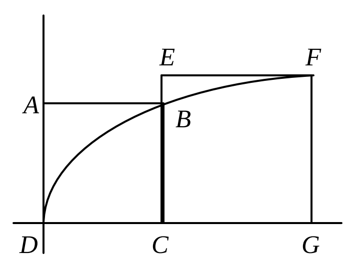

2. In isosceles $\Delta ABC$, $m\angle BAC = 108°$. $\overline{AD}$ trisects $\angle BAC$ and $BD > DC$. Find $\dfrac{BD}{DC}$.




3. A particle is moving on the curve defined by the parametric system $x = 1 - 2\cos^2 q$ and $y = \cos q$. The particle is closest to the origin when $q = \cos^{-1} a$ for $0 \le q \le \dfrac{p}{2}$. Determine the value of $a$.

4. The first term of an infinite geometric series is 2 and the sum of the series lies less than 1/10 from 2. Find the values that the common ratio $r$ can take on assuming that $r \neq 0$.

5. $ABCD$ and $EFGC$ are squares and the curve $y = k\sqrt{x}$ passes through the origin $D$ and points $B$ and $F$. Determine $\dfrac{FG}{BC}$.




6. Each side of a regular dodecagon $A_1 A_2 A_3 \ldots A_{12}$ is 2 units long. Determine the area of the pentagon $A_1 A_2 A_3 A_4 A_5$.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2005**

***Answer Sheet***

Round 1

1. (78, 100)
2. $\frac{31}{6}$
3. 70

Round 2

1. $-\frac{8}{3}$
2. 315
3. $a < -1 \text{ or } a > -\frac{1}{2}$

Round 3

1. $\frac{135}{16}$
2. $\frac{16}{9}$
3. $\frac{200}{27}$

Round 4

1. $\frac{209}{2}$
2. –1
3. $-\frac{401}{9}$

Round 5

1. $\left(3\sqrt{5}, 3\sqrt{5}\right)$
2. $\left(4, -6, 2\sqrt{2}\right)$
3. $\sqrt{33}$

Round 6

1. 19 + 22$i$
2. $\frac{1}{99}$
3. $\frac{-1+\sqrt{5}}{2}$

Team

1. 4.8
2. $\frac{1+\sqrt{5}}{2}$
3. $\frac{\sqrt{6}}{4}$
4. $-\frac{1}{19} < r < \frac{1}{21}$
5. $\frac{1+\sqrt{5}}{2}$
6. $5+2\sqrt{3}$

**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2005 - SOLUTIONS**

## Round 1 Arithmetic and Number Theory

1. So far she has 322 points. She needs a total between 400 and 445 inclusive. However, the most on one test is 100.

2. $\frac{7}{2}+\frac{5}{3}=\frac{31}{6}$

3. For $n = 17$ to 33, the ratios increase from 8:1 to 16:1 by 1's at every other number. From $n = 34$ to $n = 50$, the ratios increase from 17:2 to 25:2 by halves at every other number. From $n = 51$ to 67, the ratios increase from 25:3 to 33:3 by thirds at every other number. Thus, our first solution must occur in the 4$^{\text{th}}$ interval of 17's. At $n = 68$ the ratio is 34:4 and at $n = 70$, the ratio is 35:4 $= 8\frac{3}{4}$, making $n = \boxed{70}$.

## Round 2 Algebra 1

1. $5^{2(3x-2)} = 5^{-3(4-x)} \rightarrow 6x-4=-12+3x$

2. Let $n$ = number of nickels. $n^2+15=8n$ has solutions 3 and 5. 5 doesn't satisfy the condition. So there are 3 nickels and 12 quarters for \$3.15.

3. $x \oplus (a \oplus x) = x \oplus (ax + x) = ax^2 + x^2 + ax + x = -\frac{1}{8}$ ® $8(a+1)x^2 + 8(a+1)x + 1 = 0$.

   By the discriminant: $\left(8(a+1)\right)^2 - 4 \cdot 8(a+1) > 0 \rightarrow (a+1)(2a+1) > 0$.

   $\therefore \boxed{a < -1 \text{ or } a > -\frac{1}{2}}$.

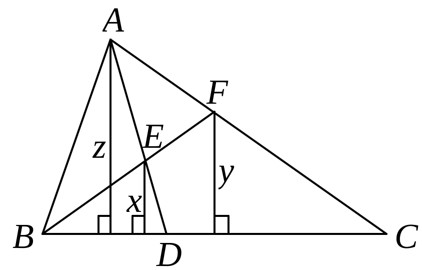
## Round 3 – Geometry

1. Denote the segments on the diameter as $3x$ and $5x$, getting $3x + 5x = 12 \rightarrow x = 1.5$. Let ED = y.
   $\frac{9}{2}\cdot\frac{15}{2} = 4y \rightarrow y = \frac{135}{16}$

2. Consider the area of the trapezoid to be the difference of the areas of two right triangles.

$$\frac{1}{2}\cdot 8\cdot\frac{10}{3}-\frac{1}{2}\cdot\frac{16}{3}\cdot\frac{20}{9}$$

3. Since $\overline{BF}$ bisects $\angle ABC$ and $AB = 2,\ BC = 3$, then $\frac{AB}{BC}=\frac{AF}{FC}=\frac{EF}{BE}=\frac{2}{3}$. Drop perpendiculars with length $x$, $y$, and $z$ from points $E$, $F$, and $A$ respectively. By similar triangles, $\frac{x}{y}=\frac{3}{5}$ and $\frac{y}{z}=\frac{3}{5} \Rightarrow \frac{x}{z}=\frac{9}{25} \Rightarrow$ $\frac{ED}{DA}=\frac{9}{25} \Rightarrow \frac{AE}{ED}=\frac{16}{9}$.




**Round 4 – Algebra 2**

1. $\frac{1}{2}=4r^3 \rightarrow r=\frac{1}{2} \rightarrow a_1 = 4\cdot 2^3 = 32$. $a_1=\frac{1}{3}(3)^4 = 27$; $\frac{27}{1-\frac{1}{3}}+\frac{32}{1-\frac{1}{2}} = \frac{81}{2}+64=\frac{209}{2}$.

2. From $f(x) = 2x+1$ we obtain $f^{-1}(x)=\frac{x-1}{2}$. Thus, $f\left(\frac{1}{f^{-1}(x)}\right) = f\left(\frac{2}{x-1}\right) =$ $\frac{4}{x-1}+1=\frac{x+3}{x-1}$. Also, $f^{-1}\left(\frac{1}{f(x)}\right) = f^{-1}\left(\frac{1}{2x+1}\right) = \frac{\frac{1}{2x+1}-1}{2} = \frac{-x}{2x+1}$. Thus, $\frac{x+3}{x-1}=\frac{-x}{2x+1} \rightarrow x^2+2x+1=0 \rightarrow x = \boxed{-1}$.

3. Note that $44^2 = 1936$ and $45^2 = 2025$. If $44 \le x < 45$, then $x[x] = 2005 \rightarrow x = \frac{2005}{44} = 45\frac{25}{44}$ and that is greater than 45. If $-45 \le x < -44$, then $[x] = -45$ and we have $x[x] = 2005 \rightarrow x = \frac{2005}{-45} = -44\frac{25}{45}$ and that works. Ans: $-\frac{2005}{45} =$ $\boxed{-\frac{401}{9} = -44\frac{5}{9}}$.

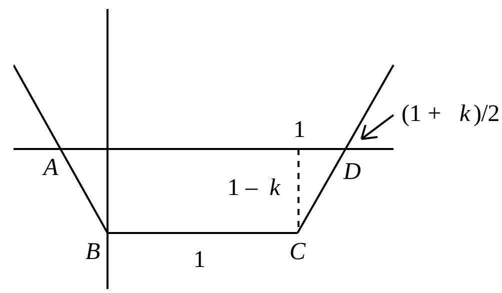
**Round 5 – Analytic Geometry**

1. Given $B(a,2a)$ and $AB=5$, then $a^2+(2a)^2=5^2 \to B(\sqrt{5},2\sqrt{5})$. Similarly, we have $D(2\sqrt{5},\sqrt{5})$. Thus, we have $C(3\sqrt{5},3\sqrt{5})$.

2. The equation of the hyperbola is also $(x-4)(y+6)=4$. It is a rectangular hyperbola centered at (4, -6) with vertices at (6, -4) and (2, -8). The distance between these is $4\sqrt{2}$, so the radius of the circle is $2\sqrt{2}$.

3. $-16 = 1(1-k) + 2\cdot\frac{1}{2}(1-k)\left(\frac{1+k}{2}-1\right) \to$

$k^2 = 33 \to k = \boxed{\sqrt{33}}$.




**Round 6 – Trig and Complex Numbers**

1. $\frac{63-16i}{1-2i}\cdot\frac{1+2i}{1+2i}=\frac{95+110i}{5}=19+22i$

2. $\frac{\tan a+\frac{2}{3}}{1-\frac{2}{3}\tan a}=\frac{7}{3}\to\frac{3\tan a+2}{3-2\tan a}=\frac{7}{3}\to\tan a=\frac{15}{23}$. $\therefore\ \tan(b-a)=\frac{\frac{2}{3}-\frac{15}{23}}{1+\frac{2}{3}\cdot\frac{15}{23}}=\frac{1}{99}$

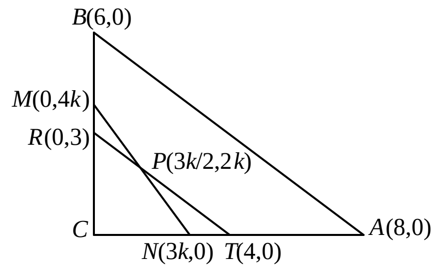
3. $\frac{\cos(2x)+1}{\cos(2x)-1}=\frac{(2\cos^2 x-1)+1}{(1-2\sin^2 x)-1}=-\frac{\cos^2 x}{\sin^2 x}$. Setting $-\frac{\cos^2 x}{\sin^2 x}=-\sec^2 x$ yields

$\cos^4 x+\cos^2 x-1=0 \to \cos^2 x = \boxed{\frac{-1+\sqrt{5}}{2}}$.

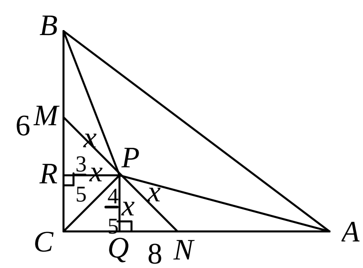
# NEW ENGLAND PLAYOFFS – 2005 - SOLUTIONS

## Team Round

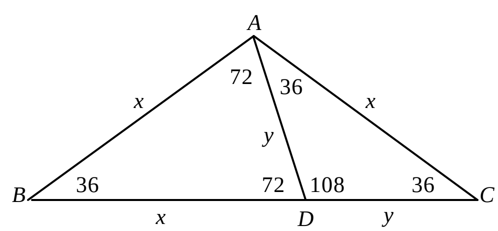
1. Place the triangle in a coordinate axis system with $C$ at the origin. $P$ must lie on the midline of $\Delta BCA$ and given $\Delta BCA \sim \Delta NCM$, let $M = (0,4k)$ and $N = (3k,0)$. Thus, $P\left(\frac{3k}{2}, 2k\right)$ must satisfy the equation for the midline $\overline{RT}$ which is $y = -\frac{3}{4}x + 3$ giving $2k = -\frac{3}{4}\left(\frac{3k}{2}\right) + 3 \;\rightarrow\; k = \frac{24}{25}$. Since $MN = 5k$, $MN = \boxed{\frac{24}{5} = 4.8}$.




Alternative solution: Draw $\overline{CP}$ and altitudes $\overline{PQ}$ and $\overline{PR}$ from $P$ to $\overline{AC}$ and $\overline{BC}$. Since the area of $\Delta ABP$ = 12 $\rightarrow$ area of $\Delta ACP$ + area of $\Delta BCP$ = 12. Let $MN = 2x$ and since $\Delta PQN$ and $\Delta PRN$ are 3-4-5 $\rightarrow PQ = \frac{4}{5}x$ and $PR = \frac{3}{5}x \rightarrow \left(\frac{1}{2}\right)(8)\left(\frac{4}{5}x\right) + \left(\frac{1}{2}\right)(6)\left(\frac{3}{5}x\right) = 12$
$\rightarrow\; 5x = 12 \rightarrow 2x = 4.8$




2. From $\Delta ABC$, $\frac{x+y}{\sin 108} = \frac{x}{\sin 36}$ and from $\Delta BDC$, $\frac{x}{\sin 108} = \frac{y}{\sin 36}$. Thus, $\frac{x+y}{x} = \frac{x}{y}$.
This gives $x^2 - xy + y^2 = 0 \;\rightarrow$
$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0 \;\rightarrow\; \frac{x}{y} = \frac{BD}{DC} = \boxed{\frac{1+\sqrt{5}}{2}}$.

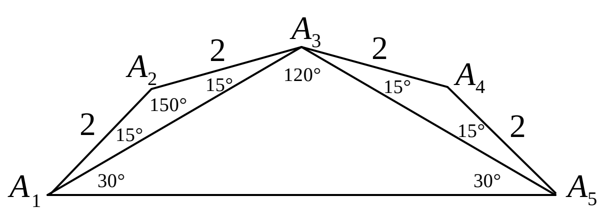


3. Let the point $P(x, y)$ lie on the curve. The distance from $P$ to the origin equals $\sqrt{x^2 + y^2}$ $= \sqrt{\left(1 - 2\cos^2 q\right)^2 + \cos^2 q} = \sqrt{4\cos^4 q - 3\cos^2 q + 1}$. The expression under the radical is a quadratic in $\cos^2 q$ and reaches its minimum at $\cos^2 q = -\frac{-3}{2 \cdot 4} = \frac{3}{8}$. Thus, the minimum occurs at $\cos q = \sqrt{\frac{3}{8}}$. Thus, $a = \sqrt{\frac{3}{8}} = \boxed{\frac{\sqrt{6}}{4}}$.

4. $\left|\frac{2}{1-r} - 2\right| < \frac{1}{10} \rightarrow -\frac{1}{10} < \frac{2r}{1-r} < \frac{1}{10} \rightarrow \frac{r-1}{10} < 2r$ and $2r < \frac{1-r}{10}$. Note that $1 - r > 0$ since otherwise the series would not converge. Solving separately, we obtain $\boxed{-\frac{1}{19} < r < \frac{1}{21}}$. Note: the restriction $r \neq 0$ need not be stated.

5. Let $BC = a$ and $FG = b$. Then, from point $B(a, a)$ we obtain $a = k\sqrt{a} \rightarrow k = \sqrt{a}$. From point $F(a + b, b)$ we obtain $b = k\sqrt{a+b} = \sqrt{a(a+b)}$. Thus, $b^2 = a^2 + ab \rightarrow$ $\left(\frac{b}{a}\right)^2 - \frac{b}{a} - 1 = 0 \rightarrow \frac{b}{a} = \boxed{\frac{1+\sqrt{5}}{2}}$.

6. Each interior angle is 150° and so the area of each of $\Delta A_1 A_2 A_3$ and $\Delta A_3 A_4 A_5$ is $\frac{1}{2} \cdot 2 \cdot 2 \cdot \sin 150° = 1$. Let $A_1 A_3 = x$.




Then $x^2 = 2^2 + 2^2 - 2 \cdot 2 \cdot 2\cos 150° = 8 + 4\sqrt{3}$. The area of $\Delta A_1 A_3 A_5 = \frac{1}{2}x^2 \sin 120°$ $= 3 + 2\sqrt{3}$. Thus, the total area is $\boxed{5 + 2\sqrt{3}}$.