THE NEW ENGLAND ASSOCIATIONS OF MATHEMATICS LEAGUES

NEAML

☆ ☆ ☆

32nd *ANNUAL MATH COMPETITION*

April 30, 2004

CANTON HIGH SCHOOL

☆ ☆ ☆

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Round 1 Arithmetic and Number Theory

1. ______________________

2. ______________________

3. ______________________

1. If $N = 9{,}998^2 + 4(9{,}998)$, determine the number of digits in N.

2. The number $1 - 0.66_9$ is what number in base 3.

3. Given n is a positive integer, $n \leq 2004$, n equals the sum of 3 consecutive positive integers, and n equals the sum of 4 consecutive positive integers. How many different values are there for n?

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Round 2   Algebra 1

1. ________________________

2. ________________________

3. ________________________

1. $\dfrac{x^{-3/2}\cdot\sqrt[5]{y}}{x^{-7/3}\cdot\sqrt[10]{y}}$ can be written as $\sqrt[n]{x^a y^b}$ where a and b are integers. What is the minimum possible sum of a, b, and n?

2. There are 24 students in a classroom. Six move from the left side of the classroom to the right side and now the right side has as many students as the left side used to have. How many did the left side have originally?

3. Given $2^{\pi}x+\left(2^{\pi}+5\right)y=3^{\sqrt{2}}x+\left(3^{\sqrt{2}}+5\right)y$, determine the value of $\dfrac{x}{y}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Round 3 – Geometry

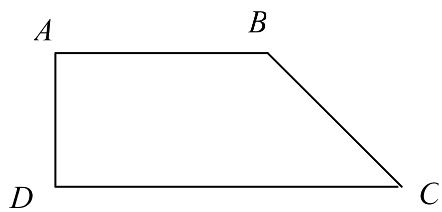
1. ____________________

2. ____________________

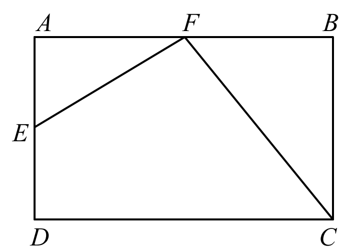
3. ____________________

Diagrams are not drawn to scale.

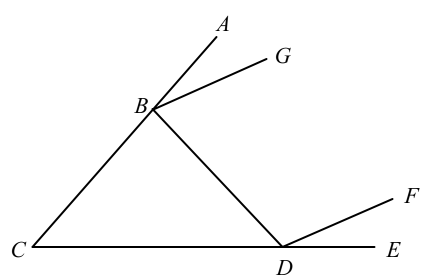
1. In quadrilateral $ABCD$, $m\angle D = m\angle A = 90°$ and $m\angle C = 45°$. If $AB = 17$ and $AD = 12$, determine the number of square units in the area of $ABCD$.



2. $ABCD$ is a rectangle, E and F are midpoints of $\overline{AD}$ and $\overline{AB}$ respectively. If $EF = 2\sqrt{3}$ and $FC = \sqrt{13}$, compute $\dfrac{AB}{AD}$.



3. If $m\angle ABG = \dfrac{1}{6} m\angle ABD$, $m\angle FDE = \dfrac{1}{6} m\angle BDE$ and $\overline{BG} \| \overline{DF}$, find the number of degrees in the measure of $\angle C$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Round 4 – Algebra 2

1. ________________________

2. ________________________

3. ________________________

1. For a, $b > 0$, compute the minimum value of $\left(\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}\right)^3$.

2. Let $[\,x\,]$ = the greatest integer less than or equal to x. Determine the domain of the function $y = \sqrt{1 - \left[x^2\right]}$. Write your answer in inequality form.

3. Solve for real x: $\sqrt{\frac{x-1}{x}} > \sqrt{\frac{x}{x-1}}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Round 5 – Analytic Geometry

1. ________________________

2. ________________________

3. ________________________

1. Line ℓ passes through the origin and divides square $ABCD$ into two regions of equal area. Given $A(7, 11)$, $B(8, 11)$, $C(8, 10)$ and $D(7, 10)$, determine the slope of ℓ.

2. Point B is in the first quadrant and lies on the line $y = 7x + 4$. Given $A(0, 4)$, $C(0, 0)$, the area of $\Delta ABC = m$ square units, and the slope of $\overline{BC} = m$, find the coordinates of B.

3. Given $O(0, 0)$, $A(0,6)$, $B(k, 0)$ for $k > 0$, if P lies on the positive x-axis so that $\overline{AP}$ is a bisector of $\angle OAB$ and P is a trisection point of $\overline{OB}$, determine the slope of $\overline{AP}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Round 6 – Trig and Complex Numbers

1. _________________________

2. _________________________

3. _________________________

1. Determine the value of $\cos(127.5°)\cos(7.5°) - \cos(37.5°)\sin(187.5°)$.

2. Determine the value of $i + 2i^2 + 3i^3 + \ldots + 2004i^{2004} + \frac{2005}{i} + \frac{2006}{i^2} + \ldots + \frac{4008}{i^{2004}}$.

3. If $\frac{\sin 4x}{1 + \cos 4x} = f(y)$ where $y = \tan x$, determine a formula for $f(y)$ in terms of y.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

Team Round

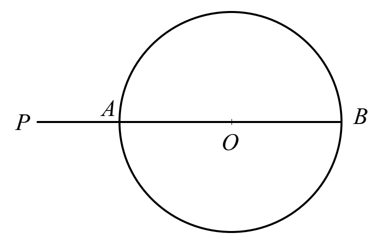
1. ________________________ **4.** ____________________

2. __(______,______)_______ **5.** ____________________

3. ________________________ **6.** ____________________

1. Points A and B are on circle O such that $\overline{AO} \perp \overline{OC}$, and $AC = 6$. Point B is on minor arc $\overset{\frown}{AC}$. Determine the minimum value of the area interior to quarter circle AOC and exterior to quadrilateral ABCO.

2. Given $\overline{PAOB}$ with A and B lying on circle O, PA and PB are the roots of $x^2 - 8x + 13 = 0$. Determine the ordered pair (a, b) where a is the length of the tangent to the circle from P and b is the area of the circle.



3. If $\sin x - \cos x = n$, determine $\sin^3 x - \cos^3 x$ in terms of n.

4. If $S = 1 - 2 - 3 + 4 + 5 - 6 - 7 + 8 + 9 - 10 - 11 + 12 + 13 - 14 - 15 + \ldots$, then let $S_1 = 1,\ S_2 = 1 - 2,\ S_3 = 1 - 2 - 3,\ S_4 = 1 - 2 - 3 + 4,\ S_5 = 1 - 2 - 3 + 4 + 5$, and so on. Find n so that $S_n = -1$ for the 2004th time. The series S has these two properties: taking the absolute value of each term produces the set of positive integers and the terms are signed in consecutive groups of 4 beginning with +, –, –, +.

5. Find the solution (x, y) to the following system if $a_i = 2a_{i-1} + 3$ and $a_1 > 0$ for $i \in \{2,3,4,5,6\}$

$$a_1 x + a_2 y = a_3$$
$$a_4 x + a_5 y = a_6$$

6. Determine all ordered pairs (x, y) that satisfy the following system on the set of real numbers:

$$\begin{cases} x + x|y| = 16 \\ x + y|x| = -9 \end{cases}$$

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2004

Answer Sheet

Round 1

1. 8
2. $.021_3$ (or .021 with base 3 implied)
3. 166

Round 2

1. 15
2. 58
3. –1

Round 3

1. 276
2. $\sqrt{35}$
3. 36 (36°)

Round 4

1. 33
2. $-\sqrt{2} < x < \sqrt{2}$
3. $x < 0$

Round 5

1. $\frac{7}{5}$
2. (4, 32)
3. $-\sqrt{3}$

Round 6

1. $-\frac{1}{2}$
2. $f(y) = \dfrac{2y}{1-y^2}$
3. 167

Team

1. $9\left(\frac{\pi}{2} - \sqrt{2}\right)$ or equivalent
2. $\frac{492}{25}$
3. $n\left(\frac{3-n^2}{2}\right)$ $\left(\text{or } \frac{3n-n^3}{2} \text{ or } \frac{3}{2}n - \frac{1}{2}n^3\right)$
4. 8014
5. (-2, 3)
6. $\left(\frac{7}{2}, -\frac{25}{7}\right)$

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2004 – Solutions

Round 1

1. $N = \left(9{,}998^2 + 4 \cdot 9{,}998 + 4\right) - 4 = \left(9{,}998 + 2\right)^2 - 4 = 10{,}000^2 - 4 =$ $\left(10^4\right)^2 - 4 = 10^8 - 4$. Since 10^8 has 9 digits, N has 8 digits.

2. $1 - 0.66_9 = 1 - \left(\frac{6}{9} + \frac{6}{81}\right) = 1 - \left(\frac{2}{3} + \frac{2}{27}\right) = 1 - 0.202_3 = 0.021_3$

3. $n = 3x + 3 = 4y + 6$ for positive integers x and y. Therefore $4y = 3(x-1)$. Since $4y + 6 \le 2004 \Rightarrow y \le 499.5$, then the possible values for y are 3, 6, 9, … 498, which are 166 possibilities.

Round 2

1. $$\frac{x^{-3/2} \cdot \sqrt[5]{y}}{x^{-7/3} \cdot \sqrt[10]{y}} = x^{5/6} \cdot y^{1/10} = \sqrt[30]{x^{25} y^3}$$

2. Let x = the original number of students on the left side and $24 - x$ be the original number on the right side. If 6 students move from left to right then the right side has $30 - x$. Thus, $x = 30 - x$ so $x = 15$.

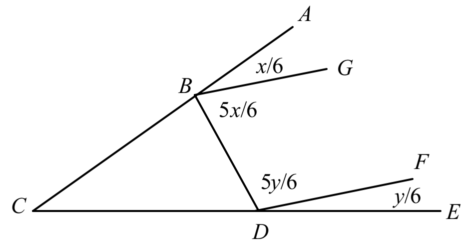
3. Let $2^{\pi} = a$ and $3^{\sqrt{2}} = b$. Then $ax + (a+5)y = bx + (b+5)y$
$ax + (a+5)y = bx + (b+5)y$ gives $(a-b)x = \left[(b+5) - (a+5)\right]y = (b-a)y$. Then $\frac{x}{y} = -1$.

Round 3

1. Draw $\overline{BE} \perp \overline{DC}$, then BE = 12 EC = 12, DC = 29

2. Let $AB = 2x$ and $AD = 2y$. Using ΔAEF, $x^2 + y^2 = \left(2\sqrt{3}\right)^2 = 12$. Using ΔFBC, $x^2 + (2y)^2 = \left(\sqrt{13}\right)^2 \rightarrow x^2 + 4y^2 = 13$. Subtracting the first from the second gives $3y^2 = 1 \rightarrow y = \frac{1}{\sqrt{3}}$. Then $x^2 + \frac{1}{3} = 12 \rightarrow x = \sqrt{\frac{35}{3}}$. Hence, $\frac{x}{y} = \frac{AB}{AD} = \sqrt{35}$.

3. Let $m\angle ABD = x \to m\angle GBD = \dfrac{5x}{6}$. Let $m\angle BDE = y \to m\angle BDF = \dfrac{5y}{6}$. Since $\overline{BG} \| \overline{DF}$, then $\dfrac{5x}{6} + \dfrac{5y}{6} = 180$ giving $x + y = 216$. Since $m\angle CBD = 180 - x$ and $m\angle BDC = 180 - y$, then $m\angle C = 180 - (180 - x + 180 - y) = x + y - 180 = 216 - 180 = 36$.

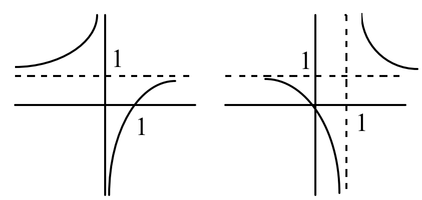


Round 4

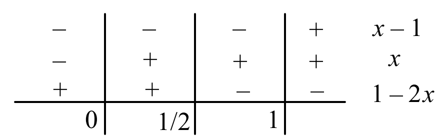
1. $M\!:\!N = 3\!:\!8 \to N = \dfrac{8}{3}M$. $M^2 - 3N = M \to M^2 - 8M = M$

2. For $\sqrt{1-\left[x^2\right]}$ to be real, $\left[x^2\right] \le 1 \to \left[x^2\right] = 0,1 \to 0 \le x^2 < 2 \to -\sqrt{2} < x < \sqrt{2}$.

3. Graphing $y = \dfrac{x-1}{x}$ and $y = \dfrac{x}{x-1}$ shows us that the domain of the inequality is $x < 0$ or $x > 1$. It is also shows that the solution set is $x < 0$.



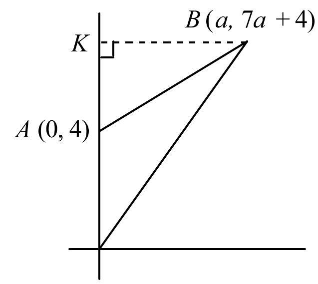
<u>Alternate solution</u>: Squaring both sides we obtain: $\dfrac{x-1}{x} > \dfrac{x}{x-1} \to \dfrac{x-1}{x} - \dfrac{x}{x-1} > 0 \to \dfrac{1-2x}{x(x-1)} > 0$. Solving using the sign graph we obtain $x < 0$ or $1/2 < x < 1$, but reject the latter set since it violates the domain of the inequality.



Round 5

1. A line passing through the center of a square bisects the square. The center of $ABCD$ is $M\left(\dfrac{15}{2}, \dfrac{21}{2}\right)$ and the slope of ℓ is $\dfrac{21/2-0}{15/2-0} = \dfrac{7}{5}$.

2. Let the coordinates of B be $(a, 7a+4)$. Then



$\frac{1}{2}\cdot AC\cdot BK = slope \rightarrow \frac{1}{2}\cdot 4a = \frac{7a+4}{a} \quad \rightarrow$

$2a^2 - 7a - 4 = 0 \ \rightarrow \ a = 4$ gives $B(4, 32)$.

3. By the Triangle Angle Bisector Theorem, if $\overline{AP}$ bisects $\angle OAB$ then $\frac{OA}{AB} = \frac{OP}{PB}$.

Since P is a trisection point then $\frac{OP}{PB} = \frac{1}{2} \rightarrow \frac{OA}{AB} = \frac{1}{2}$. Thus, ΔOAB is a 30-60-90 right triangle making $AB = 12$ and $OB = 6\sqrt{3}$. Hence, the x-coordinate of P is $2\sqrt{3}$, making the slope of $\overline{AP}$ equal $\frac{6-0}{0-2\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$.

<u>Round 6</u>

1. $\cos(127.5)\cos(7.5) - \cos(37.5)\sin(187.5) = -\sin(37.5)\cos(7.5) - \cos(37.5)[-\sin(7.5)]$

$= -(\sin 37.5\cos 7.5 - \cos 37.5\sin 7.5) = -\sin 30 = -\frac{1}{2}$.

2. Group the first 2004 terms by 4's obtaining

$\left(i + 2i^2 + 3i^3 + 4i^4\right) + \cdots + \left(2001^{2001} + 2002^{2002} + 2003^{2003} + 2004^{2004}\right) =$

$\left(i - 2 - 3i + 4\right) + \cdots + \left(2001 - 2002 - 2003 + 2004\right) = \left(-2i + 2\right) + \cdots + \left(-2i + 2\right)$.

There are 501 such terms with a sum total of $-1002i + 1002$. Grouping the next 2004 terms in the same way we obtain 501 terms of $2i + 2$ for a total of $1002i + 1002$. The sum of both groups is 2004.

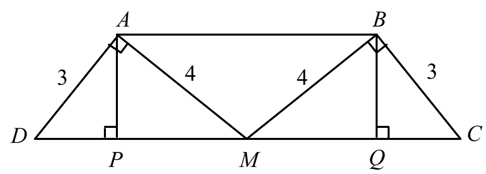
3. $\frac{\sin 4x}{1+\cos 4x} = \frac{2\sin 2x\cos 2x}{1+2\cos^2 2x - 1} = \frac{\sin 2x}{\cos 2x} = \tan 2x = \frac{2\tan x}{1-\tan^2 x}$. Thus, $f(y) = \frac{2y}{1-y^2}$.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2004 – Solutions – Team Round

1. The minimum occurs when B bisects $\overparen{AC}$. The area of ΔAOC = $\frac{1}{2}\left(3\sqrt{2}\right)^2$. The area of ΔABC is $\frac{1}{2}(6)\left(3\sqrt{2}-2\right)$.

2. Both ΔADP and ΔBOQ are 3-4-5 triangles so AD = 3 = $5x$ gives x = 3/5. Then, DP = $3x$ = 9/5 and AP = $4x$ = 12/5. The length of base $\overline{AB}$ = 10 – 2(9/5) = 32/5. The area of $ABCD$ equals $\frac{1}{2}\cdot\frac{12}{5}\left(10+\frac{32}{5}\right)=\frac{492}{25}$.



3. $\sin^3 x-\cos^3 x=\left(\sin x-\cos x\right)\left(\sin^2 x+\sin x\cos x+\cos^2 x\right)=n\left(1+\sin x\cos x\right)$. From $\left(\sin x-\cos x\right)^2=n^2$ we obtain $\sin^2 x-2\sin x\cos x+\cos^2 x=n^2\rightarrow$ $\sin x\cos x=\frac{n^2-1}{-2}=\frac{1-n^2}{2}$. Thus, $\sin^3 x-\cos^3 x$ = $n\left(1+\frac{1-n^2}{2}\right)$ = $n\left(\frac{3-n^2}{2}\right)$.

4. $S_1=1, S_2=-1, S_3=-4, S_4=0, S_5=5, S_6=-1, S_7=-8, S_8=0, S_9=9, S_{10}=-1,$ Thus, $S_{2+4k}=-1$. Starting with k = 0, The 2004th term occurs when k = 2003, giving n = 2 + 4(2003) = 8014.

5. The problem implies an invariant result. The following system leads to the answer:

$$\begin{aligned} x+5y&=13\\ 29x+61y&=125\end{aligned}$$ Solving gives x = –2 and y = 3.

More generally, starting with a_1 we obtain in turn $a_2=2a_1, a_3=4a_1+9,$. $a_4=8a_1+21, a_5=16a_1+45, and\ a_6=32a_1+93$ Using determinants, we have

$$x=\frac{\begin{vmatrix}a_3 & a_2\\ a_1 & a_5\end{vmatrix}}{\begin{vmatrix}a_1 & a_2\\ a_4 & a_5\end{vmatrix}}=\frac{(4a+9)(16a+45)-(32a+93)(2a+3)}{a(16a+45)-(2a+3)(8a+21)}=\frac{42a(a+3)}{-21a(a+3)}=-2.$$

$$y=\frac{\begin{vmatrix}a_1 & a_3\\ a_4 & a_6\end{vmatrix}}{-21a(a+3)}=\frac{-63a(a+3)}{-21a(a+3)}=3.$$

6. In the 1st quadrant the system becomes $x + xy = 16$ and $x + xy = -9$. That system has no solution. In the 2nd quadrant we have $x + xy = 16$ and $x - xy = -9$. Solving, we obtain $x = \frac{7}{2}$ which lies outside the quadrant. In the 3rd quadrant we obtain $x - xy = 16$ and $x - xy = -9$ which has no solution. Finally, in the 4th quadrant the system becomes $x - xy = 16$ and $x + xy = -9$. Solving, we obtain $x = \frac{7}{2}$ and $y = -\frac{25}{7}$ yielding the answer $\left(\frac{7}{2}, -\frac{25}{7}\right)$.