MAML

STATE INVITATIONAL MATH LEAGUE COMPETITION

March 31, 2006

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2006

Round 1 **Arithmetic and Number Theory**

1. ______________________

2. ______________________

3. ______________________

1. What is the closest multiple of 7 to the median of all 2-digit prime numbers?

2. For $n \geq 20$, determine the least integer value of n such that the product $(505)n$ has twice as many digits as n.

3. Find the largest prime factor of $320^2 - 320 + 4$.

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Round 2 **Algebra 1**

1. ______________________

2. ______________________

3. ____(______,______)______

1. Factor completely: $3x^a(x^a - 1) - 90$

2. If $\sqrt{5} + \sqrt{15} > \sqrt{a}$ for a an integer, determine the largest value of a.

3. The equation $x^2 + bx + c = 0$ has distinct roots. If 2 is subtracted from each root, the results are the reciprocals of the original roots. Determine the ordered pair (b, c).

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Round 3 – Geometry

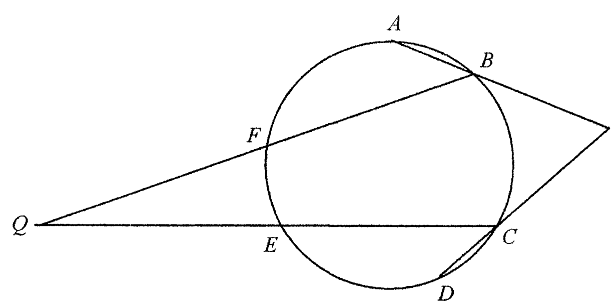
1. ______________________

2. ______________________

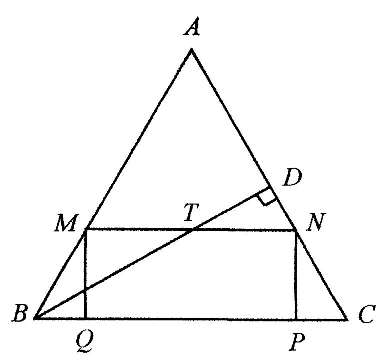
3. ______________________

1. An equilateral triangle has a common side to a square with no common interior points, thus forming a pentagon whose outer perimeter is 20. A segment is constructed between non-common vertices of the square and triangle. Determine the number of square units in the area of the smaller region so formed.

2. If $\widehat{m}AB = \widehat{m}CD = \widehat{m}EF$,
   $\widehat{m}BC = \widehat{m}DE = \widehat{m}FA$, and $\widehat{m}BC > \widehat{m}AB$,
   determine $m\angle P$.
   Note: $\widehat{m}AB$ means $m(\text{arc}\,AB)$



3. ABC is an equilateral triangle of side 2, $\overline{BD}$ is an altitude, $MNPQ$ is a rectangle, and T is the midpoint of $\overline{MN}$. Find the number of square units in the area of $MNPQ$.



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Round 4 – Algebra 2

1. $m =$ ________ $n =$ ________

2. ____________________

3. ____________________

1. If $\log_4 9 = a$ and $\log_4 36 = b$, $\log_8 \frac{216}{81}$ can be expressed in the form $ma + nb$, where m and n are simplified rational numbers. Determine the values of m and n.

2. Solve $\left|\frac{6x-6}{x+3}\right| \leq 3$. Express your answer as an inequality.

3. Find the sum of all real solutions to $\sqrt{1-\sqrt{1-x}} = x$.

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Round 5 – Analytic Geometry

1. (___ , ___ , ___)

2. (___ , ___)

3. ______________

1. Point M (2, -4) is $\frac{2}{5}$ of the way from P(-4, -8) to point Q. The equation of the line through Q perpendicular to $\overleftrightarrow{PQ}$. can be written as $ax + by = c$, where a, b, and c are relatively prime integers and $a > 0$. Determine the ordered triple (a, b, c)

2. Given $A(0, 0)$, $B(1, 2)$, and $D(13, 20)$, let C lie on $\overline{BD}$ such that the area of ΔABC is half the area of ΔACD. Determine the coordinates of C.

3. Circle O passes through the point of intersection and the focal points of $y = (x - 2)^2$ and $y = (x + 2)^2$. Determine the number of units in the radius of the circle.

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Round 6 – Trig and Complex Numbers

1. ______________________

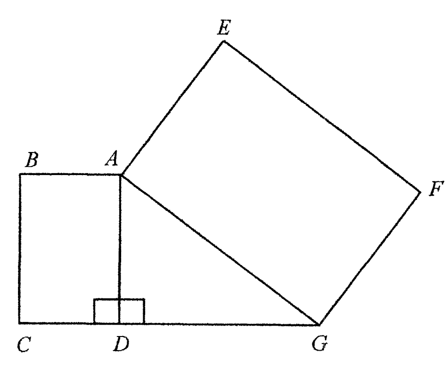
2. ______________________

3. ______________________

1. On the interval $0 \le x \le 2\pi$, how many points of intersection are there for the graphs of $y = \tan\left(x - \frac{\pi}{6}\right)$ and $y = 4\cos\left(x + \frac{\pi}{3}\right) + \frac{1}{2}$.

2. If $z_1 = 4cis\frac{\pi}{6}$ and $z_2 = 5cis\frac{\pi}{4}$, find the multiplicative inverse of $(z_1 z_2)^2$

3. In right triangle *ADG*, $AD = 6$, $DG = 8$.
   Rectangle *ABCD* ~ rectangle *AEFG*.
   If $BE = 26$, find *AE*.



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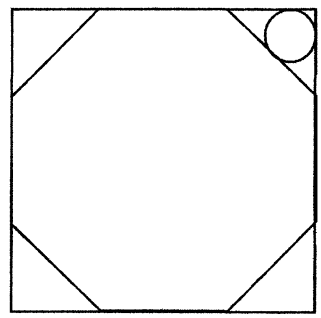
Team Round

1. ____________________ **4.** ____________________

2. ____________________ **5.** ____________________

3. ____(______,______)____ **6.** ____________________

1. A regular octagon is inscribed in a square. If the radius of the inscribed circle shown at the right is $6 - 4\sqrt{2}$ units, find the number of units in the side of the square.



2. Let S consist of the set of points (x, y) satisfying both $[x]^2 + [y]^2 = 1$ and $x^2 + y^2 \le \frac{1}{4}$ where $[x]$ is the greatest integer function. Determine the number of square units in the area of S.

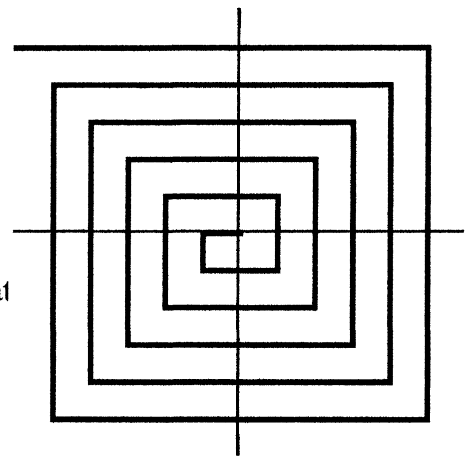
3. Congruent parabolas P_1 and P_2 have vertices $V_1(0,4)$ and $V_2(6,0)$ respectively. If the parabolas are tangent to each other and have vertical axes of symmetry, determine the point of tangency.

4. Determine the distance between point $A(-2, 4, -9)$ and the line whose equation is given parametrically by $x = 5 + 4t$, $y = 4 + t$, and $z = -1 + 3t$.

5. Given the rows of logarithm values below, where 10 is the base, determine the smallest row number such that the sum of the elements in the row exceeds 900. Note: In the table, the term between log a and log b in the row below is log (ab); assume log 2 = 0.301.

Row #											
1					log 1		log 2		log 1		
2				log 1		log 2		log 2		log 1	
3			log 1		log 2		log 4		log 2		log 1
4	log 1		log 2		log 8		log 8		log 2		log 1

6. Part of a rectilinear spiral is shown. The spiral starts at the origin and reaches the following points in succession: (–1, 0), (–1, –1), (1, –1), (1, 1), (–2, 1), (–2, –2,). It continues in this fashion, the vertical distance between two adjacent horizontal lines is 1 and the horizontal distance between two adjacent vertical lines is 1. What is the total length of the spiral when it reaches the point (–2006, –2006)?



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Answer Sheet

Round 1

1. 49
2. 199
3. 181

Round 2

1. $3(x^a - 6)(x^a + 5)$
2. 37
3. (-2, -1)

Round 3

1. 4
2. 60
3. $\frac{4\sqrt{3}}{9}$

Round 4

1. $m = -\frac{4}{3}, \quad n = 1$
2. $-\frac{1}{3} \le x \le 5$
3. $\frac{1+\sqrt{5}}{2}$

Round 5

1. $(3, 2, 37)$
2. (5, 8)
3. $\frac{289}{120}$

Round 6

1. 4
2. $\frac{1}{400} cis \frac{7\pi}{6}$
3. $5\sqrt{13}$

Team

1. 4
2. $\frac{\pi}{8}$
3. (3, 2)
4. 3
5. 13
6. 16,092,132

Solutions State Meet 2006

Round 1 Arithmetic and Number Theory:

1. Two digit primes: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

2. Since (505)(20) = 10,100 and (505)(99) = 49,995, both five-digit numbers, it is clear that n must be a three-digit number and the product must be a six-digit number. Divide 100,000 by 505 and obtain 198 plus a fraction. Then if $n = \boxed{199}$, the product will just slip beyond 100,000, namely 100,495.

3. $320^2 - 320 + 4 = 320^2 + 4 \cdot 320 + 4 - 5 \cdot 320 = (320 + 2)^2 - 1600 = 322^2 - 40^2 = (322 + 40)(322 - 40) = 362 \cdot 282 = 2 \cdot 181 \cdot 2 \cdot 141 = 2^2 \cdot 181 \cdot 3 \cdot 47$. Ans: $\boxed{181}$.

Round 2 Algebra I:

1. Multiplying and factoring out a 3: $3\left(x^{2a} - x^{a} - 30\right)$

2. $\left(\sqrt{5} + \sqrt{15}\right)^2 > a \rightarrow 20 + 10\sqrt{3} > a$. Since $\sqrt{3} \approx 1.732$, $37.32 > a$, making $a = \boxed{37}$.

3. The roots r_1 and r_2 are the two solutions to $r - 2 = \frac{1}{r} \rightarrow r^2 - 2r - 1 = 0 \rightarrow$ $(b,c) = \boxed{(-2,-1)}$.

Round 3 Geometry:

1. Area of Pentagon is $16 + 4\sqrt{3}$. Height of Pentagon is $4 + 2\sqrt{3}$. Half pentagon – right triangle = $\left(8 + 2\sqrt{3}\right) - \left(4 + 2\sqrt{3}\right) = 4$. For those who know trig, the law of sines gives a quick solution.

2. Let $m\overset{\frown}{AB} = x$ and $m\overset{\frown}{BC} = y$. Then $3x + 3y = 360° \rightarrow x + y = 120°$. Since $m\angle P = \frac{m\overset{\frown}{AD} - m\overset{\frown}{BC}}{2} = \frac{(2y + x) - y}{2} = \frac{y + x}{2} = \frac{120}{2}$, then $m\angle P = \boxed{60}$. An interesting aspect of this problem is that $m\angle Q$ is not fixed since it equals $\frac{y - x}{2}$ but it is bounded.

3. Let $NC = 2x$ making $PC = x$, $QP = 2 - 2x$, and $NP = x\sqrt{3}$, giving an area of $(2 - 2x)x\sqrt{3}$ for $MNPQ$. Since $TN = \frac{1}{2}MN$, $TN = 1 - x$ and since ΔTDN is a 30-60-90 triangle, then

$DN = \frac{1}{2}(1 - x)$. Since $DC = 1$, then $NC + ND = DC$ implies that

$2x + \frac{1}{2}(1 - x) = 1 \rightarrow 4x + 1 - x = 2 \rightarrow x = \frac{1}{3}$, making $a(MNPQ) = \boxed{\frac{4\sqrt{3}}{9}}$.

Round 4 Algebra II:

1. $\dfrac{\log \frac{216}{81}}{\log_4 8} = \dfrac{\log\ 6^3 - \log_4 3}{\frac{3}{2}} = \left(\frac{3b}{2} - \frac{4a}{2}\right) \cdot \frac{2}{3} = b - \frac{4}{3}a = \frac{3b - 4a}{3}$

2. $\left|\frac{6x - 6}{x + 3}\right| \le 3 \rightarrow \left|\frac{x - 1}{x + 3}\right| \le \frac{1}{2} \rightarrow -\frac{1}{2} \le \frac{x - 1}{x + 3} \le \frac{1}{2} \rightarrow 0 \le \frac{1}{2} + \frac{x - 1}{x + 3}$ and $\frac{x - 1}{x + 3} - \frac{1}{2} \le 0$.

   The left-hand inequality gives $\frac{3x + 1}{x + 3} \ge 0 \rightarrow x < -3$ or $x \ge -\frac{1}{3}$. The right-hand inequality gives $\frac{x - 5}{2(x + 3)} \le 0 \rightarrow -3 < x \le 5$. The intersection is $\boxed{-\frac{1}{3} \le x \le 5}$.

3. $\sqrt{1 - \sqrt{1 - x}} = x \rightarrow 1 - \sqrt{1 - x} = x^2 \rightarrow 1 - x^2 = \sqrt{1 - x} \rightarrow x^4 - 2x^2 + x = 0$. From $x\left(x^3 - 2x + 1\right) = 0$ we use synthetic division to discover that $x = 1$ is also a root giving $x(x - 1)(x^2 + x - 1) = 0$. This has roots 0, 1, $\frac{-1 + \sqrt{5}}{2}$, and $\frac{-1 - \sqrt{5}}{2}$. The last root is extraneous so the sum of the roots is $0 + 1 + \frac{-1 + \sqrt{5}}{2} = \boxed{\frac{1 + \sqrt{5}}{2}}$.

Round 5 Analytic Geometry:

1. $Q(11, 2)$.; $m\overline{PQ} = \frac{2}{3}$; $m \perp \overline{PQ} = -\frac{3}{2}$; This leads to $3x + 2y = 37$.

2. Since the triangles have the same altitude, their bases must be in a ratio of 1 to 2 so C is the trisection point of $\overline{BD}$ closest to B. Since $B = (1,2)$ and $D = (13, 20)$, then C is (5, 8)

3. From $y = \frac{1}{4p}(x - h)^2$ we know that p, the distance from the vertex to the focal point, is $p = \frac{1}{4}$.

The parabolas intersect at (0, 4), the circle passes through $A\left(-2, \frac{1}{4}\right), B(0, 4)$, and $C\left(2, \frac{1}{4}\right)$. The

5. Neglecting the first and last terms in any row, note that in any one row, say the 4^{th} row whose terms are $\log 2^1, \log 2^3, \log 2^3, \log 2^1$, the exponents are terms in Pascal's triangle. Since addition of logs results in the multiplication of the terms which gives the addition of exponents, the next row of terms, the 5^{th} in this case, will also involve exponents drawn from Pascal's triangle, namely $\log 2^1, \log 2^4, \log 2^6, \log 2^4, \log 2^1$. The sum of the terms in the *n*th row of a Pascal's triangle starting with 1 is 2^{n-1}. Here the sum will be $\log 2^{2^{n-1}}$. We want $\log 2^{2^{n-1}} > 900 \rightarrow 2^{n-1}(.301) > 900$ so $2^{n-1} > 2990$. Since $2^{11} = 2048$ and $2^{12} = 4096$, then $n - 1 > 11 \rightarrow n > 12 \rightarrow n = 13$. The sum exceeds 900 in the $\boxed{\text{13th}}$ row.

6. Note that the lengths of the segments form the sum $1 + \underline{\underline{1}} + 2 + 2 + 3 + \underline{\underline{3}} + 4 + 4 + \ldots$. Note also that the double-underlined lengths are lengths of segments that end at (–1, –1) and (–2, –2). Thus, the second segment of length 5 ends at (–3, –3), the second segment of length 7 ends at (–4, –4), . . . Hence, the second segment of length $2n - 1$ ends at $(-n, -n)$. In this case the second segment of length 4001 ends at (–2006, –2006). The length of the spiral is

$$2(1 + 2 + 3 + \text{K} + 4001) = 2 \cdot \frac{(1 + 4011)}{2} \cdot 4011 = 4012 \cdot 4011 = \boxed{16,092,132}.$$