MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2005

Round 1 **Arithmetic and Number Theory**

1. ______________________

2. ______________________

3. ______________________

1. What is the sum of the first three prime numbers having the property that they are each two more than a positive perfect cube?

2. A man has 4 gray sport jackets, 5 brown sport jackets, 3 gray pair of slacks, 2 brown pair of slacks, 6 gray ties, and 8 brown ties. Each item of one color is easily distinguishable from the others of that color. He always wears a white shirt. If each day he dresses in a sport jacket and pair of slacks and some days he wears a tie and others no tie, in how many ways can he make an outfit if he never mixes gray and brown?

3. Let N be a two-digit positive number and let M be the two-digit number formed by reversing the digits of N. If $N - M$ is a perfect cube and $N > M$, determine the number of values for N.

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Round 2 **Algebra 1**

1. ________________________

2. ________________________

3. ________________________

1. Express in simplest radical form: $\dfrac{\left(\frac{1}{2}\right)^{-3}+\sqrt[3]{\frac{27}{64}}}{\left(\frac{16}{7}\right)^{-1/2}}$.

2. On a test Sam got a certain number of answers wrong and he got exactly 60% more than that right. What is the least possible number of questions on the test, given that it is a multiple of 4? No questions were left blank.

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3. Find three consecutive positive integers such that the square of the second minus twelve times the first is three less than twice the third.

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Round 3 – Geometry

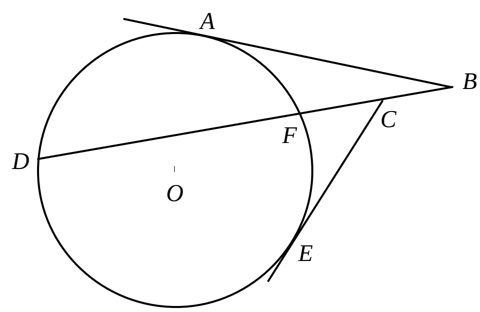
1. ________________________

2. ________________________

3. ________________________

1. Pentagon $ABCDE$ is such that if diagonal $\overline{BE}$ is drawn, it will be the base of isosceles ΔABE and $BCDE$ will be an isosceles trapezoid. If $m\angle CBA = 88^\circ$ and $m\angle C = 3m\angle CBE$, determine the number in degrees of $\angle$A.

2. $\overline{AB}$ and $\overline{CE}$ are tangent to circle O at A and E respectively. If $AB = 9\sqrt{2}$, $DF = 9$, and $CE = 6$, determine BC.



3. In a regular square pyramid, each side of the base measures 8 cm and the height is 12 cm. The top is cut off 8 cm from the base by a plane parallel to the base. What is the number of sq cm in the lateral surface area of the remaining frustum?

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Round 4 – Algebra 2

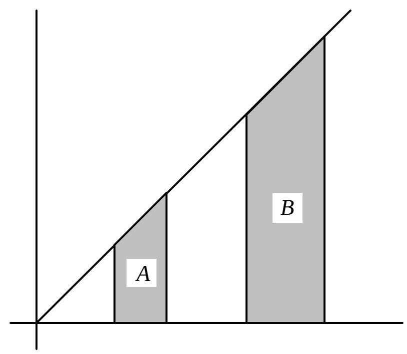
1. ________________________

2. ________________________

3. ________________________

1. In a geometric progression of positive terms, the fourth term is half the 8^{th} term and their sum is 24. The first term is of the form 2^n, where n is a real number. Determine the value of n.

2. Two trapezoids, A and B, are formed using the x-axis, the line $y = x$ and the vertical lines $x = \log_b a$, $x = \log_b a^2$, $x = \log_b a^3$, and $x = \log_b a^4$. For $a, b > 1$, determine the ratio of the area of region B to the area of A in terms of relatively prime integers.



3. Determine all values of k such that the solution set of $|x - k| < 2$ is a subset of the solution set of $\dfrac{2x-1}{x+2} < 1$. Use inequality notation to express your answer.

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Round 5 – Analytic Geometry

1. (, ,)

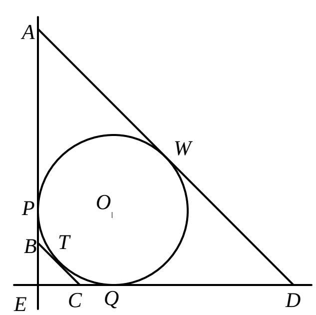
2. ____________________

3. ____________________

1. Line l passes through (-8, 5) and has a slope twice the slope of the line whose equation is $7x - 4y = 19$. If the equation of line l is written in the form $ax + by = c$ where a, b, c are relatively prime integers, with $a > 0$, determine the ordered triple (a, b, c).

2. Given the equation of the ellipse $\frac{(x-3)^2}{16} + \frac{(y+4)^2}{49} = 1$, a parabola is such that its vertex is the lowest point of the ellipse and it passes through the ends of the minor axis of the ellipse. The equation of the parabola is in the form $y = a(x-h)^2 + k$. Determine the value of a.

3. Circle O is tangent to the x- and y-axes at points Q and P respectively. $\overline{BC}$ and $\overline{AD}$ are parallel tangents to the circle with slopes of –1. If the area of $ABCD$ equals 48, and r is the radius of circle O, determine r^2.



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Round 6 – Trig and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. If $\sin^{-1}\frac{2}{3} = x$, determine $\tan 2x$. (Note: $\sin^{-1}$ is the inverse sine function.)

2. Given the sequence $a_n = (2n-1)i^n$ where $i = \sqrt{-1}$, determine the sum of the first 10 terms given that n starts at 1.

3. For $0° \le q < 360°$, determine the value(s) of q which satisfy $2(\cos q + \sin q) = \cot q + 1$

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STATE PLAYOFFS – 2005

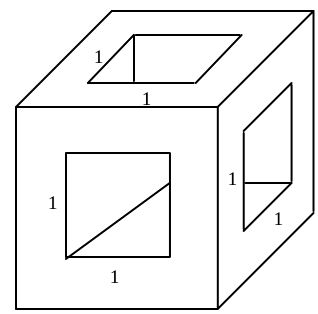
Team Round

1. ________________________ **4.** ____________________

2. ________________________ **5.** ____________________

3. ________________________ **6.** ____________________

1. Three square holes of side 1 cm are cut completely through the middle of a cube perpendicular to the faces as indicated in the diagram. If the volume of the remaining figure is 324 cm^3, find the number of cm in an edge of the cube. (The figure is not drawn to scale and does not accurately show the holes going completely through the cube)



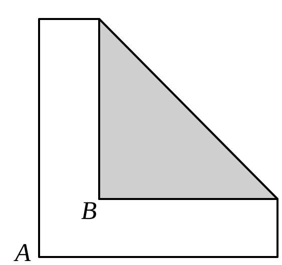
2. For acute angles A and B, if $(\tan A)(\cot B) = \frac{9}{5}$, determine the value of $\frac{\sin(A+B)}{\sin(A-B)}$.

3. From every two-digit positive integer pick the smaller digit. Numbers with the same two digits are not included since they have no smaller digit. Find the sum of all the smaller digits.

4. A farmer has 4 melons. He weighs each pair of melons and finds that 27, 41, 44, 47, and 61 are the weights of all the pairs. (Note: More than one pair may have the same weight.) Determine the weight of the heaviest melon.

5. How many isosceles triangles with integer sides and a perimeter of 2005 are there?

6. A square of side 1 unit has one side unshaded and the other side shaded. Vertex B is folded over so that B now lies on the diagonal from A of the square. If the shaded region showing has the same area as the unshaded region showing, determine, in simplified exact form, the number of units in the length of new $\overline{AB}$.



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STATE PLAYOFFS – 2005

Answer Sheet

Round 1

1. 159
2. 174
3. 6

Round 2

1. $5\sqrt{7}$
2. 52
3. 12, 13, 14

Round 3

1. 94
2. 6
3. $\frac{512\sqrt{10}}{9}$

Round 4

1. $\frac{9}{4}$
2. $\frac{7}{3}$
3. $0 \le k \le 1$

Round 5

1. 7, -2, -66
2. $\frac{7}{16}$
3. $6\sqrt{2}$

Round 6

1. $4\sqrt{5}$
2. $-11 + 9i$
3. 30, 135, 150, 315

Team

1. 7
2. $\frac{7}{2}$
3. 240
4. 32
5. 501
6. $\frac{3\sqrt{2} - 2\sqrt{3}}{3}$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2005 – Solutions

Round 1

1. 3 + 29 + 127 = 159
2. 4·3·7 + 5·2·9 = 174.
3. $10a + b - (10b + a) = 9(a - b)$. Clearly, $a - b = 3$, so we have the following ordered pairs (a, b): (4, 1), (5, 2), (6, 3), (7, 4), (8, 5), (9, 6) for a total of $\boxed{6}$ numbers.

Round 2

1. $\dfrac{8+\frac{3}{4}}{\frac{\sqrt{7}}{4}} = \dfrac{35}{\sqrt{7}} = 5\sqrt{7}$.

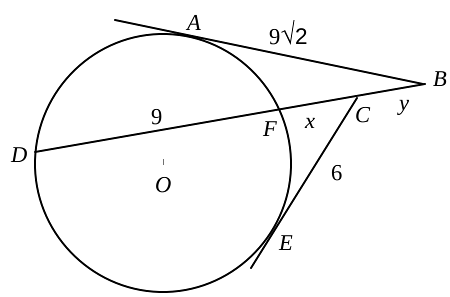
2. Let N be the number of problems and x be the number wrong. $x + \frac{8}{5}x = N \rightarrow \frac{13}{5}x = N$. The least value of x for N to be an integer is 5, $N = \boxed{13}$.

3. $(x+1)^2 - 12x = 2(x+2) - 3 \rightarrow x^2 - 12x = 0$

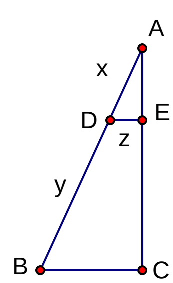
Round 3

1. m∠C = 135, m∠CBE = 45, m∠ABE =43, ∴m∠A=94.

2. Let $BC = y$ and $CF = x$, then
$AB^2 = BF \cdot BD \rightarrow \left(9\sqrt{2}\right)^2 = (x+y)(x+y+9)$ and
$CE^2 = CF \cdot CD \rightarrow 6^2 = x(x+9) \rightarrow x^2 + 9x - 36 = 0$
$\rightarrow (x+12)(x-3) = 0 \rightarrow x = 3$. Substituting into the first equation gives $y = 6$



3. In ΔACB, $\overline{AC}$ is the altitude of the pyramid and $\overline{AB}$ is the slant height. Since AC = 12, AE = 4 and BC = 4, by similar triangles, $x = \dfrac{4\sqrt{10}}{3}$, y = $\dfrac{8\sqrt{10}}{3}$, and z= $\dfrac{4}{3}$



Round 4

1. $a_8 = 16, a_4 = 8 \to r = \sqrt[4]{2} \to 8 = a_1 \sqrt[4]{8}$
2. Given $A = \frac{1}{2}(2\log_b a - \log_b a)\left(\log_b a^3\right)$ and $B = \frac{1}{2}(4\log_b a - 3\log_b a)\left(\log_b a^7\right)$, the ratio is $\frac{7}{3}$.
3. $\frac{2x-1}{x+2} - 1 < 0 \to \frac{x-3}{x+2} < 0 \to -2 < x < 3$. Since $-2 < x - k < 2 \to k - 2 < x < k + 2$, then $\boxed{0 \le k \le 1}$ makes the solution set to $|x - k| < 2$ a subset of the solution set to $\frac{2x-1}{x+2} < 1$.

Round 5

1. Slope of $l = \frac{7}{2}$. $y - 5 = \frac{7}{2}(x+8)$
2. Center of ellipse: (3, –4), Low point: (3, –11), Ends of minor axes: (-1, -4) and (7, -4). $y = a(x-3)^2 - 11$. Substitute the coordinates of an endpoint to get a.
3. Starting with $C(a,a)$, $D(b,b)$, and $O(r,r)$, then $EC = a$, $TC = \frac{a\sqrt{2}}{2}$, and since $TC = CQ$, then $EQ = r = a + \frac{a\sqrt{2}}{2} \to a = \left(2 - \sqrt{2}\right)r$. Likewise, $WE = WD = \frac{b\sqrt{2}}{2}$. Since $EO + OW = WE = r\sqrt{2} + r = \frac{b\sqrt{2}}{2}$, then $b = \left(2 + \sqrt{2}\right)r$. The area of $ABCD$ equals the difference in the areas of ΔAED and $\Delta BEQ = \frac{1}{2}b^2 - \frac{1}{2}a^2 = \frac{1}{2}\left(\left(6 + 4\sqrt{2}\right)r^2 - \left(6 - 4\sqrt{2}\right)r^2\right) = 4\sqrt{2}r^2 = 48$. Thus, $r^2 = \boxed{6\sqrt{2}}$.

Round 6

1. $\cos x = \frac{\sqrt{5}}{3}, \tan x = \frac{2\sqrt{5}}{5} \quad \tan 2x = \frac{\frac{4\sqrt{5}}{5}}{1 - \frac{4}{5}}$
2. $.i - 3 - 5i + 7 + 9i - 11 + \cdots - 19 = -11 + 9i$
3. Multiply both sides by $\sin q$, write as $f(q)$=0 and factor, getting

   $(2\sin q - 1)(\cos q + \sin q = 0)$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2005 – Solutions Continued

Team Round

1. Let x be the length of the cube's edge. The volume of each hole is $1 \cdot 1 \cdot x$, so the volume of material removed is $3x$, but that counts the little cube that is the intersection of the three holes three times. So the volume removed is $3x - 2$. Thus, $x^3 - (3x - 2) = 324 \rightarrow$ $x^3 - 3x - 322 = 0 \rightarrow (x - 7)(x^2 + 7x + 46) = 0$. The only real solution is $x = \boxed{7}$.

2. $\tan A \cdot \cot B = \frac{9}{5} \rightarrow \frac{\sin A}{\cos A} \cdot \frac{\cos B}{\sin B} = \frac{9}{5} \rightarrow \sin A \cdot \cos B = \frac{9}{5} \cos A \cdot \sin B$. Since $\frac{\sin(A+B)}{\sin(A-B)} =$ $\frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B} = \frac{\frac{9}{5}\cos A \sin B + \cos A \sin B}{\frac{9}{5}\cos A \sin B - \cos A \sin B} = \frac{\frac{9}{5}+1}{\frac{9}{5}-1}$, the answer is $\frac{7}{2}$.

3. From 10 to 19 there is one 0 and eight 1's for a total of $8 \cdot 1 = 8$. From 20 to 29, there is one 0, one 1, and seven 2's for a total of $1 + 7 \cdot 2 = 15$. From 30 to 39 there is one 0, one 1, one 2, and six 3's for a total of $1 + 2 + 3 \cdot 6 = 21$. From 40 to 49 we have $1 + 2 + 3 + 5 \cdot 4 = 26$. For 50 – 59: $1 + 2 + 3 + 4 + 4 \cdot 5 = 30$, for 60 – 69: $1 + 2 + 3 + 4 + 5 + 3 \cdot 6 = 33$, for 70 – 79: $1 + 2 + 3 + 4 + 5 + 6 + 2 \cdot 7 = 35$, for 80 – 89: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 1 \cdot 8 = 36$, and for 90 – 99 we also have 36. The sum of all the smallest digits is $8 + 15 + 21 + 26 + 30 + 33 + 35 + 36 + 36 = \boxed{240}$.

4. From ${}_4C_2 = 6$ we are expecting to see 6 different weights but there are only 5 different weights. If two of the weights were the same, then there would be only 4 different weights, so the conclusion is that two different pairs give the same result. Call the weights a, b, c, and d with $a < b < c < d$. Then we can know for sure that $a + b = 27$, $a + c = 41$, and $c + d = 61$. Subtracting the third from the second we obtain $a + 20 = d$. Now either $a + d = 44$ or $a + d = 47$. Trying $a + d = 44$ gives $2a + 20 = 44 \rightarrow a = 12$, making $d = 32$, and $b = 15$, $c = 29$. If $a + d = 47$, then $2a + 20 = 47 \rightarrow a = \frac{27}{2}$ but that makes $b = \frac{27}{2}$ which equals a and that can't be. Thus, the heaviest weight is $\boxed{32}$.

   Alternate solution: from $a + b = 27$ and $c + d = 61$, we obtain $(a + d) + (b + c) = 88$, making $a + d = b + c = 44$. From $a + d = 44$, $b + d = 47$, $c + d = 61$ we add to obtain $a + b + c + 3d = 152$. Subtract $a + b + c + d = 88$ and obtain $2d = 64$.

5. Given sides $x, x, 2005-2x$, we have $2x > 2005-2x \rightarrow x > 501.25$ and $x+2005-2x > x$; $x + 2005 - 2x > x \rightarrow x < 1002.5$. Thus, the equal sides can take on integer values from 5 02 to 1002 inclusive, a total of $\boxed{501}$ values.

6. Using the figure as marked: $2(1 \cdot x) - x^2 = \frac{1}{2}(1-x)^2$, $3x^2 - 6x + 1 = 0 \rightarrow x = \frac{3-\sqrt{6}}{3}$.

$AB = x\sqrt{2} \rightarrow AB = \boxed{\frac{3\sqrt{2} - 2\sqrt{3}}{3}}$.