MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2002

Round 1 Arithmetic and Number Theory

1. ______________________

2. ______________________

3. ______________________

1. Determine the square root of the positive difference between the palindromic integers closest to but different from 2002.

2. Find the sum of all numbers of the form $.\overline{AB}$ where A and B are decimal digits which are not necessarily distinct.

3. Let S be the set of two–digit primes. If a number is chosen at random from S, what is the probability that the product of its digits is a prime number.

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Round 2   Algebra 1

1. ________________________

2. ________________________

3. ________________________

1. The value of $\left(8^{-2/3} + 9^{-3/2}\right)^{-1}$ can be expressed in the form a/b where a and b are relatively prime integers. What is the value of $a - b$?

2. Factor completely: $4a^2 - 12ac + 9c^2 - 4^3$.

3. On a normal floor Romeo and Juliet walk at 3 feet per second. Starting at both ends of a straight moving walkway at an airport, which travels at a constant rate of k ft/sec, they walk towards each other, meeting at a point one-seventh of the way from one end of the walkway. Determine k.

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Round 3 – Geometry

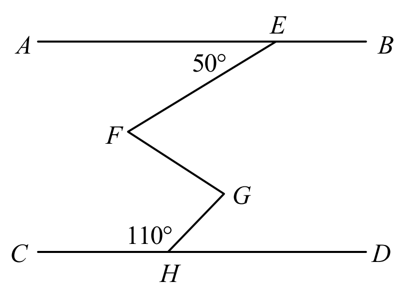
1. ________________________

2. ________________________

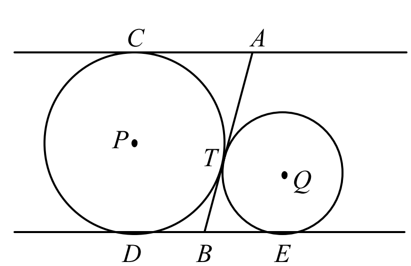
3. ________________________

1. In ΔABC, if D and E are trisection points of $\overline{AB}$ and $\overline{AC}$ closest to B and C respectively, determine the ratio of the area of ΔDEC to the area of ΔDBC.

2. If $\overline{AB}$ is parallel to $\overline{CD}$, $m\angle AEF = 50°$, $m\angle CHG = 110°$, and $m\angle EFG$ and $m\angle FGH$ are both acute and have integer measures, determine the largest possible number of degrees in the measure of $\angle EFG$.



3. $\overline{AC} \parallel \overline{DE}$, circles P and Q are tangent to $\overleftrightarrow{DB}$ at D and E respectively, circle P is tangent to $\overline{AC}$ at C, $\overline{AT}$ is the common internal tangent, and points B, T, and A are collinear. If the radii of P and Q are 10 and 8 respectively, compute TB.



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STATE PLAYOFFS – 2002

Round 4 – Algebra 2

1. ________________________

2. ________________________

3. ________________________

1. Find four consecutive odd positive integers such that the square of the first is 16 more than nine times the sum of the last three.

2. Determine the set of real x–values for which y is real in $x^3 - xy + y^2 = 0$.

3. To measure the amount of moisture in a room, Leyla created a scale with base 10 logs. If A is the amount measured by the sogginess detector, Leyla rounded down the value of $-\log A$ to the greatest integer, then multiplied the reciprocal of that integer by 100. Determine all values of A which would yield a scaled value of 25.

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STATE PLAYOFFS – 2002

Round 5 – Analytic Geometry

1. ________________________

2. ________________________

3. ________________________

1. Given $A(-2, 6)$ and $B(1, 12)$, the equation of the line perpendicular to $\overline{AB}$ and passing through the trisection point of line AB closest to A can be written in the form $ax + by = c$, where a, b, and c are relatively prime integers with $a > 0$. Determine the values of a, b, and c and write your answer as an ordered triple in that order.

2. Lines L_1 and L_2 pass through $P(1, 2)$. The slope of L_2 is twice that of L_1 and the y–intercept of L_2 is three times that of L_1. Determine the slope of L_1.

3. Let $f(x) = mx + b$ with $f(1) = 3$ and $f(0) \cdot f^{-1}(0) = -4$. Determine all possible values of b.

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Round 6 – Trig and Complex Numbers

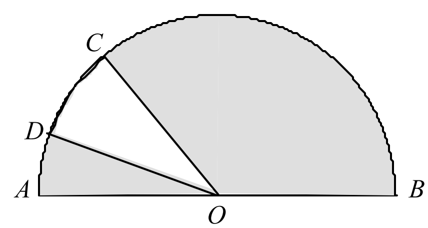
1. ______________________

2. ______________________

3. ______________________

1. Determine, in radians, the value of $\text{Tan}^{-1}(-1) + \text{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

2. Shown is semicircle O of radius 1. The area of the shaded region is five times the area of the unshaded region. The length of $\overline{CD}$ can be expressed in the form $\sqrt{a-\sqrt{b}}$ where a and b are relatively prime integers. Determine the ordered pair (a, b).



3. The expression $\left|\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{2001}-\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{2001}\right|$ can be written in the form $\sqrt{a+\sqrt{b}}$.
Determine the ordered pair (a, b).

Note: $|a+bi|=\sqrt{a^2+b^2}$.

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STATE PLAYOFFS – 2002

Team Round

1. ________________________ **4.** ____________________

2. _________________________ **5.** ____________________

3. ________________________ **6.** ____________________

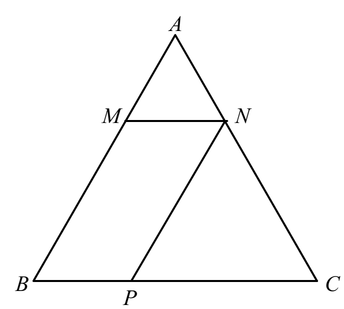
1. In (1) $x^2 + 2x + c = 0$ and (2) $x^2 + 2x + n = 0$, let c and n be distinct non-zero integers. The equations have real roots. If the discriminant of equation #1 is nine times the discriminant of #2, determine the largest value of the sum $c + n$.

2. Let n be an integer between 0 and 103 inclusive. Suppose Bob scores n points out of 103 on a test. His teacher divides n by 103, multiples the result by 100 and then rounds to obtain an integral scaled score in the interval [0, 100]. The teacher rounds down all scores with a fractional part less than or equal to 1/2 and rounds up the rest. Let p be the scaled score. Find the minimum and maximum values of n such that $n - p = 1$. Write your answer as an ordered pair in that order.

3. Find in term of a the value of $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2 - 4an + a^2 + 4n - 2a}$.

4. From the point $P(3, 0)$ a tangent line with positive slope is drawn to the graph of $\dfrac{x^2}{4} + y^2 = 1$. Determine the slope.

5. ΔABC is equilateral, M and N lie on $\overline{AB}$ and $\overline{AC}$ respectively so that $\overline{MN}$ is parallel to $\overline{BC}$. P lies on $\overline{BC}$ so that $\overline{NP}$ is parallel to $\overline{AB}$. If the area of ΔAMN plus the area of ΔNPC equals 2002, determine the greatest lower bound and least upper bound of the area of ΔABC.



6. If $\sin x$, $\sin^2 2x$, and $(\cos x)(\sin 4x)$ form an increasing geometric sequence, find the numerical value of $\cos 2x$.

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STATE PLAYOFFS – 2002

Answer Sheet

Round 1

1. 11
2. 50
3. 4/21

Round 2

1. 77
2. (2a – 3c + 8)(2a – 3c – 8)
3. 15/7

Round 3

1. 2:3 or $\frac{2}{3}$
2. 69°
3. $4\sqrt{5}$

Round 4

1. 31, 33, 35, 37
2. $x \le \frac{1}{4}$
3. $10^{-5} < A \le 10^{-4}$

Round 5

1. (1, 2, 15)
2. 4
3. –6 or 2

Round 6

1. $-\frac{7\pi}{12}$
2. (2, 3)
3. (2, 2)

Team

1. –18
2. (18, 51)
3. $\frac{1}{2-a}$
4. $\frac{\sqrt{5}}{5}$
5. min = 2002, max = 4004
6. $\sqrt{2}$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2002 - Solutions

Round 1 Arithmetic and Number Theory

1. $2112 - 1991 = 121, \quad \sqrt{121} = 11$

2. Since $.\overline{AB} = \dfrac{AB}{99}$, the problem reduces to finding this sum: $\dfrac{1}{99}+\dfrac{2}{99}+\cdots+\dfrac{99}{99}$. This equals

$$\frac{\frac{(1+99)99}{2}}{99} = 50.$$

3. $S = \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$. Of these 21 numbers, only the product of the digits of 13, 17, 31, and 71 yield a prime. Answer: 4/21.

Round 2 Algebra 1

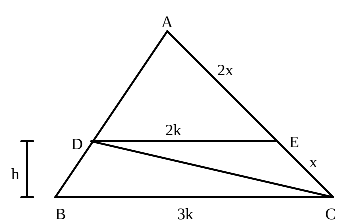
1. $\left(\dfrac{1}{4}+\dfrac{1}{27}\right)^{-1} = \left(\dfrac{31}{108}\right)^{-1} = \dfrac{108}{31} \qquad 108 - 31 = 77$

2. $4^3 = 64 \qquad (2a - 3c)^2 - 64 = (2a - 3c + 8)(2a - 3c - 8)$

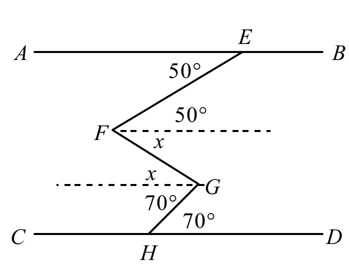
3. If Romeo is walking in the direction the walkway is moving, his rate is $3 + k$ while Juliet's is $3 - k$. Meeting at a point 1/7 from the end means that Romeo has walked six times as far. $\therefore (3 + k)t = 6(3 - k)t$. Since the t's cancel, $3 + k = 18 - 6k$ making $k = 15/7$.

Round 3 – Geometry

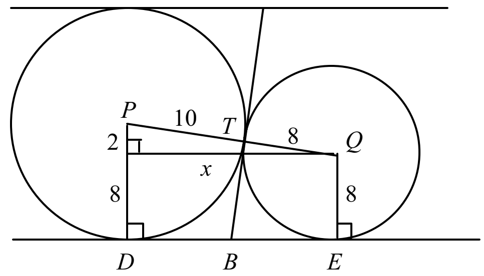
1. Since ΔDEC and ΔBDC have the same height, the ratio of areas is the ratio of their bases, $\overline{DE}$ and $\overline{BC}$ which is 2 : 3.



2. Draw parallels to $\overleftrightarrow{AB}$ through F and G. Then $m\angle EFG = 50° + x$ and $m\angle FGH = 70° + x$. Since both are acute, $70° + x < 90°$, so $x < 20°$, making the largest measure of $\angle EFG = 69°$.



3. From the diagram, $x^2 + 2^2 = 18^2$ making $x = \sqrt{320} = 8\sqrt{5}$. Since $DB = BE = BT$, then $BT = (1/2)x = 4\sqrt{5}$.



Round 4 – Algebra 2

1. Let x, $x + 2$, $x + 4$, and $x + 6$ represent the four consecutive odd integers. Then
$x^2 = 9(3x + 12) + 16 \quad \Rightarrow \quad x^2 - 27x - 124 = 0 \quad \Rightarrow \quad x = 31$ or $x = -4$ Reject -4

2. Consider $y^2 - xy + x^3 = 0$ as a quadratic in y. Solving for y we obtain: $y = \dfrac{x \pm \sqrt{x^2 - 4x^3}}{2}$. If y is to be real, then $x^2 - 4x^3 \geq 0 \rightarrow x^2(1 - 4x) \geq 0$. Thus, $x \leq 1/4$.

3. $\dfrac{100}{-\log A} = 25 \rightarrow [-\log A] = 4 \rightarrow 4 \leq -\log A < 5 \rightarrow -5 < \log A \leq -4 \rightarrow 10^{-5} < A \leq 10^{-4}$.

Round 5 – Analytic Geometry

1. The trisection point is (-1, 8). $y - 8 = (-1/2)(x + 1)$

2. Using $y = mx + b$ for L_1 and $y = 2mx + 3b$ for L_2, we have $2 = m\cdot 1 + b$ and $2 = 2m\cdot 1 + 3b$. Solving simultaneously gives $b = -2$.

3. Since $f(1) = 3$, $m + b = 3$. Clearly, $f(0) = b$. Since $f^{-1}(x) = \dfrac{x-b}{m}$ then $f^{-1}(0) = \dfrac{-b}{m}$. From $f(0)f^{-1}(0) = -4$ we have $b\left(\dfrac{-b}{m}\right) = -4 \rightarrow \dfrac{b^2}{4} = m$. Substituting into $m + b = 3$ we obtain $\dfrac{b^2}{4} + b = 3$ $\rightarrow b^2 + 4b - 12 = 0 \rightarrow (b + 6)(b - 2) = 0$ so $b = -6$ or 2.

Round 6 – Trig and Complex Numbers

1. $(-\pi/4) + (-\pi/3) = (-7\pi/12)$

2. Since sector DOC is one–sixth the circle, $m\angle DOC = 30°$. Thus, $CD^2 = 1^2 + 1^2 - 2(1)(1)\cos 30° \rightarrow CD^2 = 2 - \sqrt{3} \rightarrow CD = \sqrt{2 - \sqrt{3}}$.

3. $(\cos 30 + i \sin 30)^{2001} = (\cos 30 + i\sin 30)^{12(166)}(\cos 30 + i \sin 30)^9 = 1(\cos 270 + i \sin 270) = -i$.
Similarly, $(\cos 45 + i\sin 45)^{8(250)}(\cos 45 + i\sin 45)^1 = 1(\cos 45 + i \sin 45) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$. Thus,

$$\left|-i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 - i^2\left(\frac{\sqrt{2}}{2} - 1\right)^2} = \sqrt{2 + \sqrt{2}}.$$

Team Round

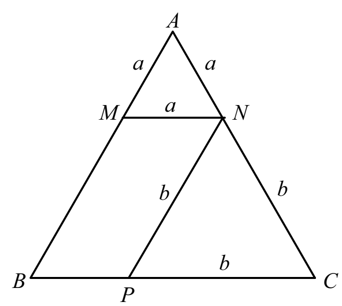
1. $9(4-4n) = 4-4c \rightarrow 9n-c = 8$. Solutions aren't distinct if $n = c = 1$ and solutions aren't real if n, c > 1. Thus, let $n = -1$, then $c = -17$ and $n+c = -18$.

2. If $n-p = 1$, then $\frac{1}{2} < n - \frac{100n}{103} \leq \frac{3}{2} \rightarrow \frac{103}{6} < n \leq \frac{309}{6} \rightarrow 17.16 < n \quad 51.5$.
Thus, $n \in \{18, 19, 20, \ldots 50, 51\}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2-4an+a^2+4n-2a} = \sum_{n=1}^{\infty} \frac{2}{(2n-a)^2+2(2n-a)} = \sum_{n=1}^{\infty} \frac{2}{(2n-a)(2n-a+2)} =$
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-a} - \frac{1}{2n-a+2} = \frac{1}{2-a} - \frac{1}{4-a} + \frac{1}{4-a} - \frac{1}{6-a} + \frac{1}{6-a} - \frac{1}{8-a} + \ldots$, which is a telescoping series which converges to $\frac{1}{2-a}$.

4. Let the slope of the line tangent to the ellipse be m. The equation of that line is $y = m(x-3)$. By substitution we have $\frac{x^2}{4} + (mx-3m)^2 = 1 \rightarrow (4m^2+1)x^2 - 24m^2x + 4(9m^2-1) = 0$. For the line to be tangent this equation must have one solution in x so set the discriminant equal to 0. Thus, $\left(24m^2\right)^2 - 4^2(4m^2+1)(9m^2-1) = 0$ giving $576m^4 - (574m^4 + 80m^2 - 16) = 0 \rightarrow 5m^2 = 1$. Thus, $m = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

5. Since $\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} = k$, then $a^2 + b^2 = \frac{4k}{\sqrt{3}}$. The area of
$\Delta ABC = \frac{(a+b)^2\sqrt{3}}{4} = \left(a^2+2ab+b^2\right)\frac{\sqrt{3}}{4} =$
$\left(\frac{4k}{\sqrt{3}} + 2ab\right)\frac{\sqrt{3}}{4} = k + (ab)\frac{\sqrt{3}}{2}$. To obtain the greatest



lower bound for the area of ΔABC let a or b tend toward 0 and the area is clearly greater than k. Using the arithmetic mean–geometric mean inequality, $\frac{a^2+b^2}{2} > \sqrt{a^2b^2} = ab$, then the largest ab can be is $\frac{2k}{\sqrt{3}}$. The maximum area is $k + \frac{2k}{\sqrt{3}}\frac{\sqrt{3}}{2} = 2k$. Thus, $k <$ area $\Delta ABC \quad 2k$. Here, $2002 <$ area $\Delta ABC \quad 4004$.

6. $(\sin^2 2x)^2 = (\sin x)(\cos x \cdot \sin 4x) = \sin x \cdot \cos x \cdot 2(2 \sin x \cos x)\cos 2x = 4 \sin^2 x \cos^2 x \,)\cos 2x$.
$\sin^4 2x = (\sin 2x)^2 \cos 2x$. Then dividing both sides by $\sin^2 2x$ gives
$\sin^2 2x = \cos 2x$. Replacing $\sin^2 2x$ by $1 - \cos^2 2x$ and setting equal to 0 gives
$\cos^2 2x + \cos 2x - 1 = 0 \rightarrow \cos 2x = \frac{-1+\sqrt{5}}{2}$. Now, $\sin x = \frac{1}{\sqrt{2}}\frac{\sqrt{5}-1}{2}$, $\sin^2 2x = \frac{\sqrt{5}-1}{2}$, and
$\cos x \sin 4x = \sqrt{2}\frac{\sqrt{5}-1}{2}$, and the common ratio is $\sqrt{2}$.