IMLEM Meet #5
March, 2022

Intermediate
Mathematics League
of
Eastern Massachusetts

Calculator Meet

• CALCULATORS: only *scientific calculators* allowed for meets #4 & #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

Category 1
Mystery
Meet #5 - March, 2022

Calculator Meet

1) The sum of two whole numbers is 73 and their difference is 7. What is their product?

2) What is the largest two-digit whole number that is divisible by 3 and whose digits differ by 2 ?

3) A fence that is 271.8 feet long spans two properties, one owned by Frick and the other by Frack. Frick's portion of the fence is 16.8 feet longer than Frack's. How many feet long is Frick's section of the fence?

Answers

1) ________

2) ________

3) ________

Solutions to Category 1
Mystery
Meet #5 - March, 2022

Answers	
1)	1320
2)	75
3)	144.3

1) Guessing and checking: Approximate the numbers, adding and subtracting 3 or 4 from about half of 73. About half of 73 is 36. Adding and subtracting 3 or 4 from 36 yields 33 and 40.
The product of (33)(40) = 1320.
For students who know how to convert such a problem into a system of equations,

$$X + Y = 73$$
$$X - Y = 7$$

Adding the equations results in 2X = 80 and X = 40.
Then Y = 33.

2) Large two-digit whole numbers divisible by 3 are 72, 75, 78, 81, 84, 87, 90, 93, 96, and 99. The only one of those where the two digits differ by 2 is 75.

3) Strategies similar to those of #1 can be employed. Here is shown the systems of equations approach:

$$X + Y = 271.8$$
$$X - Y = 16.8$$

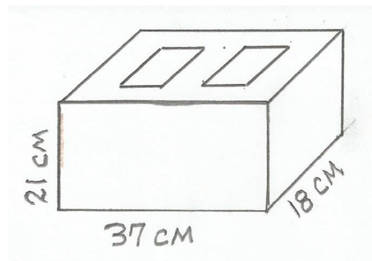
Adding the two equations produces 2X = 288.6 and X = 144.3 and Y = 127.5.
So, Frick's section is 144.3 feet long.

Category 2
Geometry
Meet #5 - March, 2022

Calculator Meet

1) Nicholas unwraps a gift contained in a cube-shaped box that has a volume of 729 cubic inches. How many square inches of wrapping paper are on the surface of the box (all sides)?

2) A rectangular cement block has two rectangular holes going completely through it. Each hole measures 13 by 15 centimeters at the top of the block. How many cubic centimeters of cement were used to make the cement block pictured?



3) It takes 21 seconds to inflate a spherical balloon to a diameter of 8 inches. How long should it take to inflate a similar balloon to a diameter of 40 inches if it is inflated at the same rate as for the smaller balloon? If the answer is B minutes and C seconds, then give the value of B if C is less than 60.

Answers

1) ________

2) ________

3) ________

Solutions to Category 2
Geometry
Meet #5 - March, 2022

Answers
1) 486
2) 5,796
3) 43

1) Take $\sqrt[3]{729}$ to find the length of one edge = 9 inches.
Each face has an area = (9)(9), or 81 square inches.
Since there are six surfaces, the surface area is 6 x 81, or 486 square inches.

2) Subtract the volume of the two holes from the volume indicated by the outer dimensions of the block:

$$(18)(21)(37) - (2)(13)(15)(21)$$
$$= 13{,}986 - 8{,}190$$
$$= 5{,}796$$

So, the amount of cement needed to make the block is 5,796 cubic centimeters.

3) The simplest approach requires this knowledge: The ratio of the volumes of two similar objects is equal to the cube of their linear ratio. (Also, while we are at it: The ratio of the surface areas of two similar objects is equal to the square of their linear ratio.)
Dividing the two diameters, 40 / 8, yields the fact that the larger balloon is five times the diameter of the smaller balloon, or 125 (five cubed) times its volume.
Multiply 21 by 125 to find the number of seconds required to inflate the larger balloon = 2625 seconds.
Divide 2625 by 60 to convert the time to minutes = 43.75 minutes = 43 minutes and 45 seconds. Since the question asks for the value of B, the number of minutes, then B = 43.
A more popular approach would be to calculate the two volumes and then divide to find how many times larger the big balloon is:

$$\frac{\frac{4}{3}\pi(20^3)}{\frac{4}{3}\pi(4^3)} = \frac{\frac{4}{3}\pi(8000)}{\frac{4}{3}\pi(64)} = \frac{8000}{64} = \mathbf{125.}$$

The remainder of the solution would be the same as for the former solution.

Category 3
Number Theory
Meet #5 - March, 2022

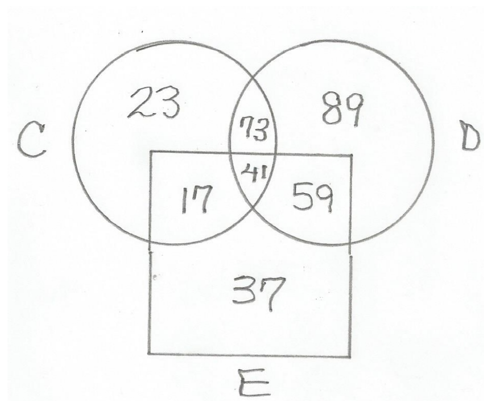
Calculator Meet

1) Set A = { all factors of 12 }
Set B = { all factors of 42 }
What is the sum of the numbers in $A \cap B$ . . . that is, the intersection of sets A and B ?

2) Circles C and D and square E intersect to form seven numbered regions. Find the sum of the numbers in the regions formed by the following:

$$(C \cap D) \cup [E \cap (C \cup D)]$$

where U means "union."



3) Of the 460 kids at the school picnic, 30% ate a hot dog and 75% ate a hamburger. Of the kids who ate a hot dog, 65 of them also ate a hamburger. How many of the 460 kids ate neither a hot dog nor a hamburger?

Answers

1) ________

2) ________

3) ________

Solutions to Category 3
Number Theory
Meet #5 - March, 2022

Answers
1) 12
2) 190
3) 42

1) A = { 1, 2, 3, 4, 6, 12 }
B = { 1, 2, 3, 6, 7, 14, 21, 42 }
The intersection of A and B is { 1, 2, 3, 6 }
Then 1 + 2 + 3 + 6 = 12.

2) $C \cap D = 73, 41$

$C \cup D = 23, 17, 73, 41, 89, 59$

$E \cap \sim C \cup D) = 17, 41, 59$

$[\ C \cap D) \cup [E \cap \sim C \cup D) = 73, 41, 17, 59$

$73 + 41 + 17 + 59 = 190$

3) The number of kids who ate hot dogs is 30% of 460, or 138. Those who ate hamburgers (burgers) is 75% of 460, or 345. The number who ate both a hot dog and a burger = 65, as given in the problem. To help fill in the blanks in the Venn diagram, those who ate just a hot dog = 138 - 65, or 73. Those who ate just a burger = 345 - 65, or 280. To calculate the number of kids who had neither a hot dog nor a burger, subtract the total number of kids who ate meat from the total number of kids who attended the picnic:

$$\begin{aligned} & 460 - (73 + 65 + 280) \\ = \ & 460 - (418) \\ = \ & 42 \end{aligned}$$

Therefore, 42 students ate neither a hot dog nor a hamburger.

Category 4 *Calculator Meet*
Arithmetic
Meet #5 - March, 2022

1) Two octahedral dice, each with faces numbered 1, 2, 3, 4, 5, 6, 7, 8 are rolled. The probability that the sum of the two numbers on the top surfaces of the dice is greater than 4 is given by the fraction A / B. What is the sum of A + B if the fraction A / B is in lowest terms?

2) If a fair coin is flipped six times, what is the probability that the result is four or more heads? If the answer is expressed as a percent, rounded to the nearest whole percent, or X percent, then what is the whole number value of X ?

3) Willy has four red, six green, and nine blue candies in his pocket. He reaches into his pocket and takes out three candies, at random. The probability that he chose candies of three different colors is P %, where P % is rounded to the nearest whole-number percent. What is the whole-number value of P ?

ANSWERS

1) ________

2) ________

3) ________

Solutions to Category 4
Arithmetic
Meet #5 - March, 2022

Answers	
1)	61
2)	34
3)	22

1) There are (8)(8), or 64 possible sums, most of them repetitive. To count the number of sums greater than 4, it may be easier to count the that are 4 or less and subtract that number from 64. Those sums are 1+1, 2+1, 1+2, 2+2, 1+3, and 3+1, or six sums of 4.
64 - 6 = 58.
So, the probability is 58/64 = 29/32 in reduced form.
Then the sum of the numerator and denominator is 29 + 32, or 61.

2) Use the "choose" or "combination" function, since there is no importance as to the order in which the heads land.

number of (4 heads) = 6C4 = $\frac{6!}{4!(6-4)!}=15$

number of (5 heads) = 6C5 = $\frac{6!}{5!(6-5)!}=6$

number of (6 heads) = 6C6 = $\frac{6!}{6!(6-6)!}=1$

These numbers can be found in the 7th row of Pascal's Triangle:
1 6 15 20 15 6 1 where the sum is 64.
The total number of possible outcomes for tossing six coins is 2^6, or 64.

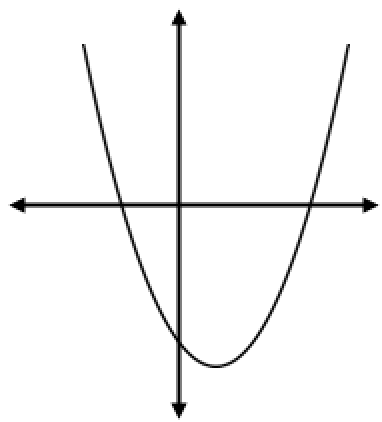
So, the probability of getting four or more heads = $\frac{15+6+1}{64}=\frac{22}{64}=0.34375.$
Rounding to the nearest whole percent yields 34%.
So, if 34 % is X %, then the whole-number value of X is 34.

3) This is an example of probability without replacement.
RGB RBG BRG BGR GRB GBR
= (4/19)(6/18)(9/17) + (4/19)(9/18)(6/17) + . . .
At this point, students may realize that all six products will have the same numerator and same denominator: (4)(6)(9) / (19)(18)(17). The sum of those six fractions is (6)(216 / 5814), or 1296 / 5814 = 0.2229 . . or 22% when rounded to the nearest whole-number percent. So, P = 22.

Category 5 *Calculator Meet*
Algebra
Meet #5 - March, 2022

1) What is the sum of all values of A for which $A^2 + 7A = 120$?

2) The graph of a parabola intersects the X-axis at (- 4, 0) and (9, 0) and intersects the Y-axis at (0, - 108).
If the parabola represents the equation $y = ax^2 + bx + c$
then what is the value of a + b + c ?
(The diagram is not drawn to scale.)



3) If an object is launched straight up into the air with an initial upward velocity of R meters per second, then its distance, D, above its starting point at time T seconds after it was launched

is $RT - 5T^2 = D$. A small rocket is launched straight up into the air from ground level with an initial upward velocity of 55 meters per second. It will attain an altitude of 90 meters and then keep rising until it reaches a maximum altitude. At that point, the rocket will fall back toward its starting point. How many seconds after the rocket reaches its maximum altitude will it again be 90 meters above its starting point?

<u>ANSWERS</u>

1) ________

2) ________

3) ________

Solutions to Category 5
Algebra
Meet #5 - April, 2022

Answers

1) - 7

2) - 120

3) 3.5

1) $A^2 + 7A = 120$
$A^2 + 7A - 120 = 0$
$(A + 15)(A - 8) = 0$
$A = -15$ or $A = 8$
So, the sum of all values of A is
$-15 + 8$
$= -7$

2) The x-intercepts allow us to write this quadratic equation in factored form: $y = a(x + 4)(x - 9)$. Using the y-intercept to substitute for x and y the values of 0 and -108, respectively, yields $-108 = a(0 + 4)(0 - 9)$ or $-108 = a(4)(-9)$.
Therefore, $-108 = a(-36)$ and $a = 3$.
Now we have $y = 3(x + 4)(x - 9)$.
Multiplying gives us $y = 3(x^2 - 5x - 36)$ or $y = 3x^2 - 15x - 108$.
$a = 3$, $b = -15$, and $c = -108$.
$a + b + c = 3 + (-15) + (-108)$ or -120.

3)

$D = RT - 5T^2$	Formula
$90 = 55T - 5T^2$	Substitute the given values
$5T^2 - 55T + 90 = 0$	Standard form
$T^2 - 11T + 18 = 0$	Divide both sides by 5
$(T - 9)(T - 2) = 0$	Factor
$T = 9$ or $T = 2$	Zero factor theorem

The rocket attains a height of 90 meters at 2 seconds after launch and again at 9 seconds. Due to the symmetry of the parabolic graph, the maximum height is attained at the average of the two times the rocket achieved the same height twice - once on the way up and again on the way down. The average of 2 and 9 is $(2 + 9) / 2$, or 5.5 seconds.
Therefore, the rocket reached an altitude of 90 meters exactly 3.5 seconds after it attained its maximum height at 5.5 seconds.

Category 6
Team Round
Meet #5 - March, 2022

Each of the following six problems is worth <u>six</u> points.

1) The sum of the squares of two consecutive positive odd integers is 12,170. What is the sum of the two consecutive odd integers?

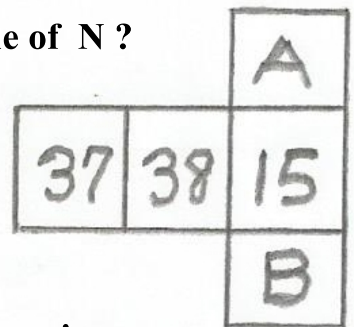
2) Simplify. Express your answer as a decimal, rounded to the nearest hundredth.

$$4+\cfrac{8}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2}}}}$$

3) In the year 1995, the United States Mint produced 13.5 billion pennies. If all of the pennies were redeemed (exchanged) for quarters, then how many quarters would be needed? If your answer is expressed in scientific notation as $A \times B^{N}$, then what is the value of $5A + 3B - 2N$?

4) If $\frac{N}{4}$ is 2 more than $\frac{N}{8}$, then what is the value of N ?

5) In the figure to the right, the sum of the numbers in the horizontal row is equal to the product of the numbers in the vertical column. What is the value of the product AB ?



6) Sam can inflate a 6-inch diameter spherical balloon in 8.3 seconds. How many seconds would it take Sam to inflate a spherical balloon that is 15 inches in diameter if he inflates it at the same rate? Round your final answer to the nearest tenth of a second.

<u>ANSWERS</u>

1) ________

2) ________

3) ________

4) ________

5) ________

6) ________

Solutions to Category 6
Team Round
Meet #5 - March, 2022

ANSWERS

1) 156

2) 9.52

3) 41

4) 16

5) 6

6) 129.7

1) One option is to write and solve a quadratic equation $X^2 + (X + 2)^2 = 12{,}170$.
A quicker, simpler approach would be to figure that is the sum of the two squares is 12,170, then one of the squares would be about half that, or about 6,085. The square root of 6,085 is about 78. The consecutive odd integers are 77 and 79. Their sum is 156. Check that the sum of their squares is 12,170.

2)
$$4+\cfrac{8}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2}}}} = 4+\cfrac{8}{1+\cfrac{1}{2+\cfrac{1}{\frac{9}{2}}}}$$

$$= 4+\cfrac{8}{1+\cfrac{1}{2+\frac{2}{9}}} \qquad 4+\cfrac{8}{1+\cfrac{1}{\frac{20}{9}}}$$

$$= 4+\cfrac{8}{1+\frac{9}{20}} \qquad 4+\cfrac{8}{\frac{29}{20}} \qquad 4+\frac{160}{29}$$

$$= \frac{276}{29} = 9.517\ldots = 9.52 \text{ rounded.}$$

3) To convert pennies to quarters, divide by 25.
13.5 billion divided by 25 is 540,000,000 or 5.4×10^8.
A = 5.4, B = 10, and N = 8.

$$\begin{aligned} & 5A + 3B - 2N \\ =\ & 5(5.4) + 3(10) - 2(8) \\ =\ & 27 + 30 - 16 \\ =\ & 41. \end{aligned}$$

4)
$$\begin{aligned} N / 4 &= 2 + N / 8 \\ 2N &= 16 + N \\ N &= 16. \end{aligned}$$

5) 37 + 38 + 15 = 15AB
90 = 15AB
6 = AB.

6) When any two 3-dinemsional figures are similar, the ratio of their volumes is the cube of the ratio of any two corresponding lengths. The two diameters are in a ratio of 15:6, or 2.5:1. So, the ratio of the two volumes is the cube of 2.5, or 15.625:1. It should take Sam 15.625 times as long to inflate the larger balloon as the smaller balloon.
(15.625)(8.3) = 129.6875 seconds.
Rounded to the nearest tenth gives us 129.7 seconds.

A student could alternatively employ the formula for the volume of a sphere as (4/3)(pi)(cube of radius) for each balloon and divide the larger volume by the smaller volume, then multiply by 8.3. Whether the student uses 3.14 for pi or some other approximation, as long as the same approximation is used for calculating each volume, the pis will "cancel" each other out.