Intermediate Mathematics League
of
Eastern Massachusetts

Calculator Meet

CLUSTER COORDINATORS - A reminder to all students of some of the rules and of appropriate behavior during this meet: • Many of you are guests in someone else's school – please be respectful of the classrooms and spaces you are using. Any "out of control" behavior in the halls or during a round is not acceptable. If an adult deems your behavior disrespectful or inappropriate, your score may not be counted. • CALCULATORS: only *scientific calculators* allowed for meets #4 & #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

1) A whole number, J, leaves a remainder of 2 when it is divided by 6. Another whole number, K, leaves a remainder of 5 when it is divided by 6. What is the remainder when the sum of J + K is divided by 6 ?

2) The figure to the right is a large square that has been subdivided by using horizontal and vertical segments into smaller squares of various sizes and non-square rectangles. How many squares are there in all?
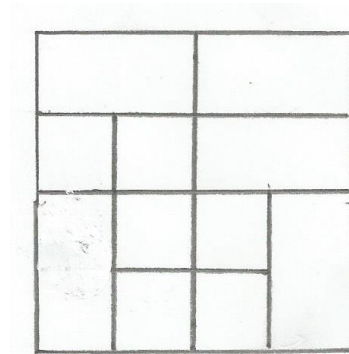
3) The product of two consecutive positive integers is 7482. What is the sum of those two consecutive positive integers?

Answers

1) _____

2) _____

3) _____

Solutions to Category 1
Mystery
Meet #4 - February, 2022

1) A student could simply choose two random numbers
   that fit the criteria, say J = 20 and K = 23, then add
   the numbers, divide by 6, and note the remainder.
   In this specific case, J + K = 20 + 23 = 43.
   Dividing 43 by 6 gives a quotient of 7 with a
   remainder of 1.
   In general, J = 6A + 2 while K = 6B + 5.
   J + K = 6A + 2 + 6B + 5 = 6A + 6B + 7
   = 6A + 6B + 6 + 1.
   Then 6A + 6B + 6 = 6(A + B + 1), so that
   6A + 6B + 6 is a multiple of 6. Then dividing
   6A + 6B + 6 + 1 by 6 leaves a remainder of 1.

Answers	
1)	1
2)	13
3)	173

2) If we consider the largest square a 4x4 and the smallest a 1x1, then the
   entire figure consists of the following:
   > one 4x4
   > one 3x3
   > five 2x2s
   > six   1x1s
   That's a total of 13 squares.

3) Each of the consecutive positive integers would be the approximate square
   root of 7482. Using the calculator, a student could easily compute the
   square root of 7482 as about 86.498 . . .   So, the two consecutive integers
   are 86 and 87. Verify that (86)(87) = 7482 . . and it does.
   So, the sum of the two consecutive integers is 86 + 87, or 173.
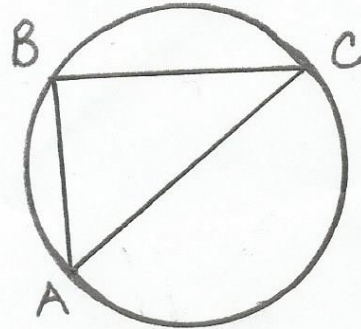
Calculator Meet

1) Using $\pi \approx 3.14,$ how many square units
   are in the area of this circle? AC is a
   diameter. B is a point on the circumference.
   AB = 16 and BC = 30.

2) The area of the larger square is 256 square inches. The midpoints of its
   four sides are connected to form a smaller square. A circle is inscribed in
   the larger square. How many square inches are in the shaded portion of
   the circle that lies outside the smaller square, as shown? Use $\pi \approx 3.142.$
   Round your final answer to the nearest
   whole number.
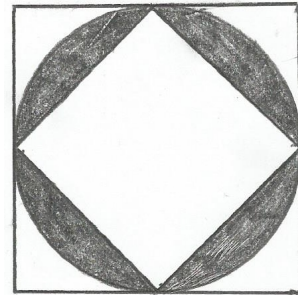
3) Line AE is tangent to the circle at point A. Angle CAE measures
   78 degrees. Angle CFB measures 47 degrees. ARC DC measures
   163 degrees. How many degrees are in the measure of minor arc AB?

Answers

1) _____ units

2) _____ sq. in.

3) _____ degrees
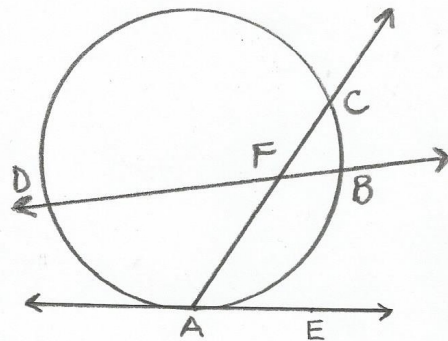
Solutions to Category 2
Geometry
Meet #4 - February, 2022

1) Use the Pythagorean Theorem to find the length of
   the diameter. Halve that number to find the length
   of the radius. Then apply the formula for the area
   of the circle.
   $16^2 + 30^2 = (AC)^2$
   $256 + 900 = (AC)^2$
   $1156 = (AC)^2$
   $34 = AC$
   Therefore, the radius of the circle is 17.
   Area = (pi)(radius)^2
       = $(3.14)(17)^2$
       = (3.14)(289)
       = 907.46

2) If the area of the larger square is 256, then one side is the square root of 256,
   or 16. Then the radius of the circle is 8. The smaller square has an area
   that is half the area of the large square, or 128, as the two diagonals divide the
   larger square into four smaller squares, half of which belong to the smaller
   square.
   The shaded area = (area of circle) - (area of smaller square)
                   = (pi)(radius)^2 - 128
                   = $(3.142)(8)^2$ - 128
                   = 201.088 - 128
                   = 73.088
                   = 73 when rounded to the nearest whole number.

3) If angle CAE = 78, then arc CBA = twice that, or 156.
   Strategy: Find the measure of arc CB, then subtract that from arc CBA
   to get arc AB.
   The two lines intersecting at point F create an angle CFB that is the average
   of arcs CB and DA. Since arc CBA = 156, then the rest of the circle's
   circumference, arc CDA = 360 - 156, or 204.
   Then arc CDA - arc CD = arc DA = 204 - 163 = 41.
   Now angle CFB = the average of arcs CB and DA, so 47 = (1/2)(41 + arc CB)
   and 94 = 41 + arc CB   and then   53 = arc CB.
   Finally, arc AB = arc CBA - arc CB
                   = 156 - 53
                   = 103.
   Therefore, minor arc AB = 103 degrees.

1) It is 4 o'clock on a 15-hour clock. After 723 hours, the power in the battery died and the clock stopped running at P o'clock. What is the value of P?

2) Jeff's son places nickels (5-cent coins) into 50 containers. He puts 17 nickels into the first container, 23 into the next, 29 into the next, and so on, adding six more nickels to the next container than he did into the previous container. After he has filled all 50 containers, how many dollars worth of nickels did Jeff's son use? Express your answer as a whole number of dollars.

3) $\sum_{K=1}^{n} K^2 = \dfrac{(n)(n+1)(2n+1)}{6}$ is the formula for adding consecutive

square numbers in the following series: $1^2 + 2^2 + 3^2 + ... + n^2$.

For example, $\sum_{K=1}^{4} K^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

We can achieve the same result by substituting 4 for n into the

formula: $\dfrac{(n)(n+1)(2n+1)}{6} = \dfrac{(4)(4+1)(2(4)+1)}{6} = \dfrac{(4)(5)(9)}{6} = \dfrac{180}{6} = 30.$

** Important - this formula works for series that begin with K = 1.

Find the value of the following sum: $81 + 100 + 121 + ... + 3969.$

Answers
1) _____
2) _____
3) _____

Solutions to Category 3
Number Theory
Meet #4 - February, 2022

1) **Divide 723 by 15, then take the remainder
   and add it to 4.**
   723 / 15 = 48 with a remainder of 3.
   Then 4 + 3 = 7.

2) **Total number of nickels = 17 + 23 + 29 + . . .**
   The value of the 50th term is (6)(50) + 11,
   or 311.
   The sum of these 50 numbers is
   (half the total number of numbers)(sum of first and last terms)
   = (1/2)(50)(17 + 311)
   = (25)(328)
   = 8200 nickels
   = $ 410.

3) **3969 is the square of 63.**
   **Subtract the sum of the first eight squares from the sum of the first
   sixty-three squares.**
   **The sum of the first eight squares can be easily accomplished without
   using the formula, but here is the formula approach:**
   (8)(8 + 1)[(2)(8) + 1] / 6 = (8)(9)(17) / 6 = 204
   The sum of the first 63 squares is
   (63)(63 + 1)[(2)(63) + 1] / 6 = (63)(64)(127) / 6 = 85,344.
   Finally, 85,344 - 204 = 85,140.

1) The sales tax rate in Massachusetts is 6.25%. What is the total cost, including tax, of a $76 jigsaw?

2) An item at the store was mistakenly reduced in price by 20%.
A stock clerk discovered the mistake and raised that lowered price by N% to restore the former price. What is the whole number value of N ?

3) The formula for compound interest is $A = P(1+r)^t$ provided that
P = the principal, or amount invested,
r = the annual (yearly) rate at which the principal is invested,
t = the amount of time, in years, that the money is invested, and
A = the total amount of money, including the principal and the interest.

Bethany found her great-grandfather's bank book in a box in the attic. His savings account on February 14, 1928 had a balance of $7.12. Assuming an average rate of 5% over the course of time, how many dollars is the savings account worth on February 14, 2022? Round your answer to the nearest dollar.

ANSWERS

1) $ _____

2) N = _____

3) $ _____

Solutions to Category 4
Arithmetic
Meet #4 - February, 2022

1) Multiply the cost of the jigsaw by 1.0625.
   The 1.0625 includes 100% of the cost of the
   jigsaw plus the 6.25% tax.
   (76)(1.0625) = 80.75.

2) The simplest solution involves selecting a cost
   that is easy to work with, say $100. Then
   reducing the price by 20% means reducing it by
   $20 down to $80. Then raising the $80 back up
   to $100 means raising the new price by $20,
   which is 25% of $80. So, N = 25.

3) The account was 94 years old in 2022 (subtracting the years 2022
   and 1928 gives 94 years).
   A = (7.12)(1 + 0.05)^94
     = (7.12)(1.05)^94
     = 698.673 . . .
   or $699 when rounded to the nearest dollar.

1) The sun cast long shadows onto the pavement in the late afternoon.
    Jules is 171 centimeters tall while Verne is 153 centimeters tall.
    Jules' shadow is 684 centimeters long. How many centimeters long
    is Verne's shadow?

2) Four chicken sandwiches and three drinks cost $20.05. Seven
    chicken sandwiches and four drinks cost $32.85. How much money
    would it cost to provide a classroom of 23 students so that each
    student receives two chicken sandwiches and one drink? Round
    your answer to the nearest whole number of dollars.

3) In Philadelphia, Pennsylvania, a set of parallel train tracks is
    400 miles long. On one end of one track, Train Theodore leaves the
    station at 5:00 PM. On the opposite end of the other track, Train
    Theresa leaves the station at 6:30 PM. If Train Theodore travels
    an average of 40 mph and Train Theresa travels an average of
    60 mph, then they meet at a time designated by A:BC, where A
    is the hour and the digits of the minutes are B and C. What is the
    sum of A + B + C?

ANSWERS

1) _____ **cm**

2) $ _____

3) _____

Solutions to Category 5
Algebra
Meet #4 - February, 2022

1) $\dfrac{\text{height}}{\text{shadow}} = \dfrac{171}{684} = \dfrac{153}{X}$

Then $171X = (684)(153)$
$171X = 104{,}652$
$X = 612$

So, Verne's shadow is 612 centimeters long.

2) Let C = the number of chicken sandwiches
D = the number of drinks.
$4C + 3D = 20.05$ and $7C + 4D = 32.85$.
To eliminate D, multiply both sides of the first equation by 4 and both sides of the second equation by - 3, then add the two equations:
$16C + 12D = 80.2$ and $- 21C - 12D = - 98.55$
Adding yields $- 5C = - 18.35$, so $C = 3.67$ and then $D = 1.79$.
So, a chicken sandwich costs $3.67 and a drink costs $1.79.
So that members of the class of 23 students each has two chicken sandwiches and one drink, the total cost will be $(23)[(2)(3.67) + 1.79] = (23)[7.34 + 1.79] = (23)(9.13) = \$209.99$.
Rounding to the nearest whole dollar yields $210.

3) Let T = the number of hours that Theresa rides.
$T + 1.5$ = the number of hours that Theodore rides.
$60T$ = the number of miles that Theresa rides.
$40(T + 1.5)$ = the number of miles that Theodore rides.

Their total distance = 400 miles, so $60T + 40(T + 1.5) = 400$
$60T + 40T + 60 = 400$
$100T + 60 = 400$
$100T = 340$
$T = 3.4$

So, 3.4 hours, or 3 hours and 24 minutes after 6:30 P.M. is 9:54 P.M.
So, $A = 9$ and $B = 5$ and $C = 4$, therefore $A + B + C = 9 + 5 + 4 = 18$.

Category 6
Team Round   *Each of the following six*
Meet #4 - February, 2022   *problems is worth <u>six</u> points.*

1) What whole number between 100 and 150 has exactly three factors?

2) Frank drove to work, averaging 120 kilometers per hour. After work, he returned home, following the same route. There was lots of traffic, so he averaged only 80 kilometers per hour. How many kilometers per hour was his average rate of speed for the entire round trip?

3) A line is tangent to two circles, as shown, at points A and B. The radii of the two circles are 25 and 10 units, respectively. AB = 36 units. How many units long is the distance between the centers of the two circles? The diagram is not to scale.
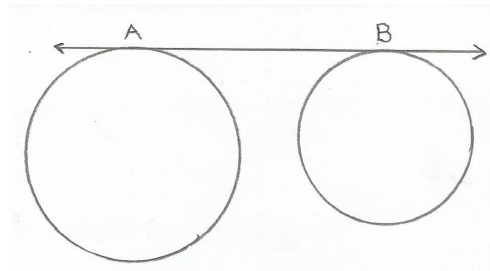
4) The product of three consecutive even numbers is 97,152. What is the sum of those three numbers?

5) If $\dfrac{4^{2021} + 4^{2022}}{5} = 2^X$ then what is the value of X ?

6) Find the value of A + B + C + E + F in this geometric sequence:
   7   A   B   C   1792   E   F   114,688

ANSWERS

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

Solutions to Category 6
Team Round
Meet #4 - February, 2022

ANSWERS	
1)	121
2)	96
3)	39
4)	138
5)	4042
6)	36,428

1) Only squares of prime numbers have exactly three factors. The only such number between 100 and 150 is the square of 11, or 121.

2) A typical error would be to average the two rates, so 100 is not the answer. The average rate for the round trip divides the total distance by the total time. Since those data are not given, a simple approach would be to choose a distance that is divisible by both 120 and 80, say 240. Then the average rate for the entire trip would be $(240 + 240) / (2 + 3)$ or $480 / 5$ or 96.

3) Draw the two radii and a segment connecting the two centers, forming a right trapezoid, as the two radii each perpendicular to the tangent at their points of contact. Then draw a segment from the center of the small circle to the radius of the larger circle that is parallel to AB. as shown. The triangle formed has bases of 15 and 36. Use the Pythagorean Theorem to find the length of the segment that connects the two centers. Or note that the bases of 15 and 36 are multiples of the bases of a 5-12-13 right triangle. Multiplying those numbers by 3 yields the triple 15-36-39. So, the segment connecting the two centers is 39 units long.
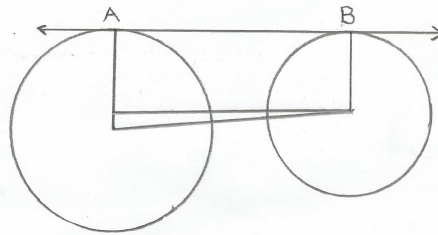
4) The middle of the three consecutive even numbers is approximately the cube root of 97,152, or about 45.97. The three even numbers are 44, 46, and 48. Their product is 97,152. So, $44 + 46 + 48 = 138$.

5) The numerator can be factored to $4^{2021}(1 + 4)$, or $4^{2021}(5)$. Dividing by 5 yields 4^{2021} or $(2^2)^{2021}$ or 2^{4042}. Therefore, $X = 4042$.

6) The common ratio: $(7)(X^4) = 1792$ so $X^4 = 256$ and $X = 4$. So, $A = 28$, $B = 112$, $C = 448$, $E = 7168$, $F = 28,672$ and sum $= 36,428$.