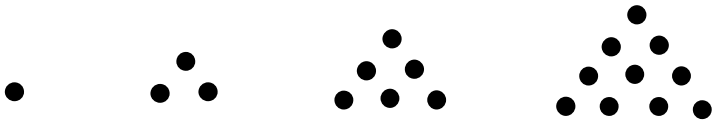
IMLEM Meet #3
January, 2018

Intermediate Mathematics League of Eastern Massachusetts

Category 1
Mystery
Meet #3 - January, 2018

1) Observe the number of dots in each of these four pictures:



If the pattern continues, then how many dots will be in the tenth picture?

2) There are two whole numbers,  **and**  **, such that**

 **+**  **= 19 and**  **X**  **= 48 .**

What is the value of  **if**  **>**  **?**

3) A rectangle is subdivided into ten squares as pictured. What is the greatest number of rectangles in this picture, either by considering individual squares that are also rectangles, or by connecting any number of adjacent squares to form longer rectangles?

Answers

1) _____

2) _____

3) _____

Note: The caricatures pictured above are of iconic rock star Elvis Presley, aka "The King," and legendary heavyweight boxer Mohammed Ali, alias Cassius Clay, aka "The Greatest." Both had birthdays in January.

Solutions to Category 1
Mystery
Meet #3 - January, 2018

1) The pattern of numbers, if continued, goes like this:
1 3 6 10 15 21 28 36 45 55
with the sequence of differences between the terms as
2 3 4 5 6 7 8 9 10
The tenth term is 55.

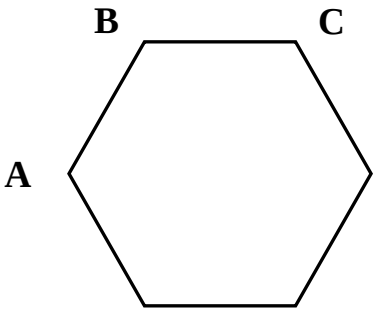
2) The two numbers whose product is 48 and whose sum is 19 are 3 and 16. The requested larger number is 16.

3) There are 10 1x1 rectangles (the 10 squares)
and 9 1x2 rectangles
and 8 1x3 rectangles
and so on, until we have 1 1x10 rectangle. Simply add the whole numbers from 1 to 10, inclusive. That sum is 55.

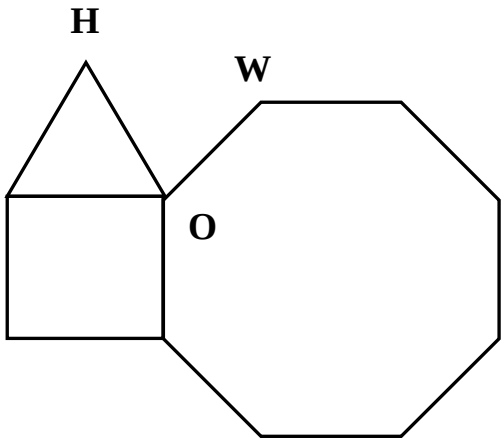
Answers	
1)	55
2)	16
3)	55

Category 2
Geometry
Meet #3 - January, 2018

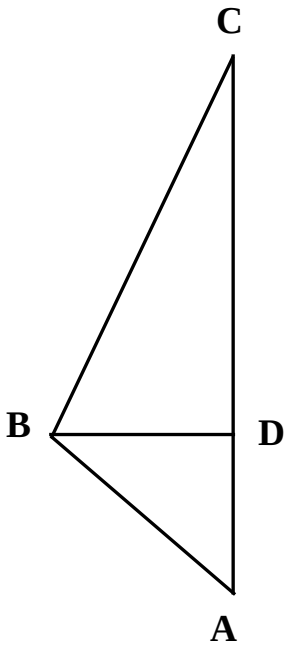
1) A diagonal of a convex polygon connects any two non-consecutive vertices No side of a polygon is a diagonal. For example, in the regular hexagon at the right, the segment AB is not a diagonal, but AC is a diagonal. How many diagonals does this regular hexagon have?



2) A square, an equilateral triangle, and a regular octagon meet at a common point, O. How many degrees are in acute angle HOW ?



3) In the diagram to the right, BD is perpendicular to AC. BD = 15. AC = 56. Point D divides AC such that AD:DC = 5:9. How many units are in the perimeter of triangle ABC?



Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #3 - January, 2018

Answers
1) 9
2) 75
3) 120

1) Pick a vertex, any vertex. I choose A. From point A can be drawn 3 diagonals. There are 3 from B. Without counting any of the diagonals twice, there are 2 from C, 1 from the next vertex, and no additional diagonals. Summing: 3 + 3 + 2 + 1 = 9.

2) Each angle of the equilateral triangle measures 60 degrees, while each angle of the square measures 90 degrees and each angle of the octagon measures 135 degrees. At point O, angle HOW measures
360 - (60 + 90 + 135)
= 360 - (285)
= 75 degrees.

3) Scale the ratio of AD:DC by a factor of four to attain AD = 20 and DC = 36. (Check: 20 + 36 = 56, the total length of AC.)
Two legs of right triangle ADB measure 20 and 15, respectively. Using the Pythagorean Theorem yields that the hypotenuse, AB, measures 25. (One could, alternatively, scale the 3-4-5 right triangle by a factor of 5 to yield a similar triangle with sides 15-20-25.) For right triangle BDC, one could use the Pythagorean Theorem to find the length of BC or else scale the 5-12-13 right triangle by a factor of 3 to obtain 15-36-39 so that BC = 39.
The perimeter of triangle ABC = AB + BC + AC = 25 + 39 + 56 = 120.

Category 3
Number Theory
Meet #3 - January, 2018

1) Convert the base 3 number 2011 to base 10.

2) Avogadro's Constant, one of the most important numbers in the scientific fields of chemistry and physics, is the number of parts, usually atoms or molecules, in one mole of a substance. That number, in scientific notation, is about 6.022×10^{23}. If Avogadro's Constant were written as a whole number, instead, then how many digits would it contain?

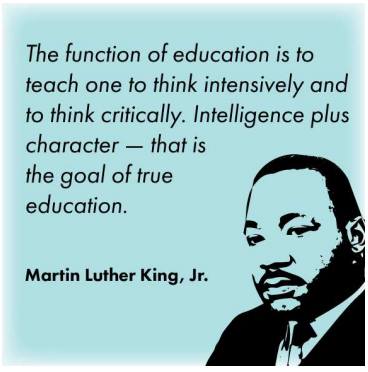
3) Convert the base 4 number 321.123 to a base 8 number. (The "point" separates the whole number part from the fractional part.)

Answers

1) _____

2) _____

3) _____



Solutions to Category 3
Number Theory
Meet #3 - January, 2018

Answers	
1)	58
2)	24
3)	71.33

1) The base 10 equivalent of 2011 (base 3) is, from right to left,
1(1) + 1(3) + 0(9) + 2(27)
= 1 + 3 + 0 + 54
= 58.

2) Moving the decimal point 23 places to the right yields the whole number that looks like 6022 followed by 20 zeroes, so that whole number has a total of 24 digits.

3) From right to left, the base 4 number 321.123 has a base 10 value of 3(1/64) + 2(1/16) + 1(1/4) + 1(1) + 2(4) + 3(16)
= 3/64 + 2/16 + 1/4 + 1 + 8 + 48
= 3/64 + 8/64 + 16/64 + 57
= 27/64 + 57.
Converting to base 8 limits the value of each digit to be less than 8. The whole number part: 57/8 = 7 with remainder 1. So, the whole number part is 71. Now for the fraction part, a bit more difficult. Express 27/64 in terms of eighths and 64ths. Since 24/64 is equivalent to 3/8, that leaves 3 64ths leftover. So, the fraction part is 0.33 and the entire base 8 value, combining the whole number part and the fraction part, is 71.33.

One could change from base 4 directly to base 8 as follows, where the caret (^) is an exponent:

3(4^2) + 2(4^1) + 1(4^0) + (1/4) + (2/16) + (3/64)
= 8*3*2 + 8 + 1 + (2/8) + (1/8) + (3/64)
= 6*8 + 1*8 + 1 + (3/8) + (3/8^2)
= 7*8 + 1 + (3/8) + (3/8^2)
= 71.33 in base 8.

Category 4
Arithmetic
Meet #3 - January, 2018

1) Evaluate: 2^5+5^2

2) What is the value of $\sqrt{\left(\sqrt[6]{\left(\sqrt{\left(\sqrt{64}\right)^2}\right)^4}\right)^3}$ **?**

2) If $\sqrt{X} + \sqrt[3]{Y} = 9$, **then find the sum of all possible values of**

Y - X if both X and Y are positive whole numbers.

ANSWERS

1) _____

2) _____

3) _____

January 1, 1776 - During the American Revolution, George Washington unveiled the Grand Union Flag, the first national flag in America (see above).

Solutions to Category 4
Arithmetic
Meet #3 - January, 2018

Answers
1) 57
2) 8
3) 1092

1) $2^5+5^2 = 32 + 25 = 57.$

2) $\sqrt{\left(\sqrt[6]{\left(\sqrt{\left(\sqrt{64}\right)^2}\right)^4}\right)^3} = \sqrt{\left(\sqrt[6]{\left(\sqrt{8^2}\right)^4}\right)^3} = \sqrt{\left(\sqrt[6]{8^4}\right)^3}$

$= \sqrt{\left(\sqrt[6]{2^{12}}\right)^3} = \sqrt{\left(4\right)^3} = \sqrt{64} = \mathbf{8.}$

3) The following ordered pairs, (X, Y), make the equation true:
(1, 512), (4, 343), (9, 216), (16, 125), (25, 64), (36, 27), (49, 8), and (64, 1).
The corresponding values of Y - X are:

512 - 1 = 511
343 - 4 = 339
216 - 9 = 207
125 - 16 = 109
64 - 25 = 39
27 - 36 = - 9
8 - 49 = - 41
1 - 64 = - 63

The sum of these differences of Y - X is

511 + 339 + 207 + 109 + 39 + (- 9) + (- 41) + (- 63)
= 1092.

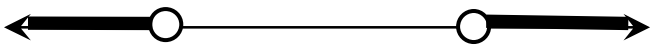
Category 5
Algebra
Meet #3 - January, 2018

1) **Find the value of this sum:** $|6| + |-13| + |0|$

2) **If** $|X+3| = 7$, **then what two values of X make this absolute value equation true?**

3) **The graph below represents all values of X that make the accompanying absolute value inequality true. What is the absolute value of A ?**

$|X-A| > 7$



- 20 **- 6**

ANSWERS

1) _____

2) ___ and ___

3) _____

Solutions to Category 5
Algebra
Meet #3 - January, 2018

Answers

1) 19

2) 4 and - 10

3) 13

1) $|6| + |-13| + |0| = 6 + 13 + 0 = 19.$

2) X = 4 or X = - 10.

3) The midpoint of - 20 and - 6 is their average, or, - 13. The points graphed on the number line are more than 7 units from - 13. So, A = - 13 and its absolute value is 13.

Category 6
Team Round
Meet #3 - January, 2018

Each of the following nine problems is worth <u>four</u> points.

1) **The diagonal of a rectangle is 30 inches long and its length is 24 inches. How many square inches are in the area of the rectangle?**

2) **Convert the base 10 numeral 406 to a base 7 numeral.**

3) **A regular N-sided polygon has an exterior angle of 12 degrees. How many cm are in the perimeter of this polygon if one side is 11 cm long?**

4) **What is the sum of the two values of W that make this absolute value equation true?** $|15 - 3W| = 204$

5) **What is the quotient when the sum of the odd prime factors of 7854 is divided by the sum of its even prime factors?**

6) **What is the greatest possible positive difference, in square centimeters, between the areas of two rectangles if the perimeter of each rectangle is 168 cm and all lengths and widths are positive whole numbers?**

<u>ANSWERS</u>

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

7) _____

8) _____

9) _____

7) **How many whole numbers are there between** $\sqrt[5]{300}$ **and** $\sqrt[3]{800}$

8) **What is the units digit (ones place) of** 39^7 **?**

9) **Aleia worked for 40 hours and earned $120. Kaija earned twice as much per hour as Aleia. How many dollars did Kaija earn by working 27 hours?**

Solutions to Category 6
Team Round
Meet #3 - January, 2018

ANSWERS	
1)	432
2)	1120
3)	330
4)	10
5)	19
6)	1681
7)	6
8)	9
9)	162

1) Using the Pythagorean Theorem, $A^2+B^2=C^2$, $A^2+24^2=30^2$ . . . $A^2+576=900$ . . . $A^2=324$ . . . and A = 18. The rectangle has dimensions 18 x 24 and area of 432 sq. in.

2) Divide 406 by 343 to get 1 with remainder 63. Divide 63 by 49 to get 1 with remainder 14. Divide 14 by 7 to get 2 with no remainder. Therefore, the base 7 equivalent is 1120.

3) 360 / 12 = 30, so the polygon has 30 sides. The perimeter is (30)(11) = 330 cm.

4) 15 - 3W = 204 or 15 - 3W = - 204.
3W = - 189 or 3W = 219.
W = - 63 or W = 73. The sum is 10.

5) 7854 = 2 x 3 x 7 x 11 x 17. The sum of the odd factors is 3 + 7 + 11 + 17 = 38. The sum of the even factors is 2, since there is only one even factor. The quotient of the two sums is 38 / 2, or 19.

6) The two extremes of areas are the 1 x 83 and the 42 x 42, or 83 and 1764 sq cm, respectively. The difference is 1764 - 83 = 1681.

7) Students must use number sense in order to realize that the fifth root of 300 is more than 3 and the cube root of 800 is less than 10. The whole numbers in between are 4, 5, 6, 7, 8, and 9, or a total of six.

8) One only need to multiply out the units digits! The first seven powers of 39 have a units digits of 9, 1, 9, 1, 9, 1 and 9, respectively. So, the seventh power of 39 has a units digit of 9.

9) Aleia earned 120 / 40, or $3 per hour. Kaija earned twice that rate, or $6 per hour. Kaija earned a total of (6)(27), or $162 for working 27 hours.