Intermediate Mathematics League of Eastern Massachusetts

This is a calculator meet!

1) What is the average of the four different prime factors of 1365 ?

2) The ratio of my current age to my age six years ago is 5 : 3 . How many years old will I be in 17 years ?

3) If the expression $10^{40} - 57$ (ten to the 40th power minus 57) is written as an integer in base ten, then what is the sum of the digits of that integer ?

Answers

1) _____

2) _____

3) _____
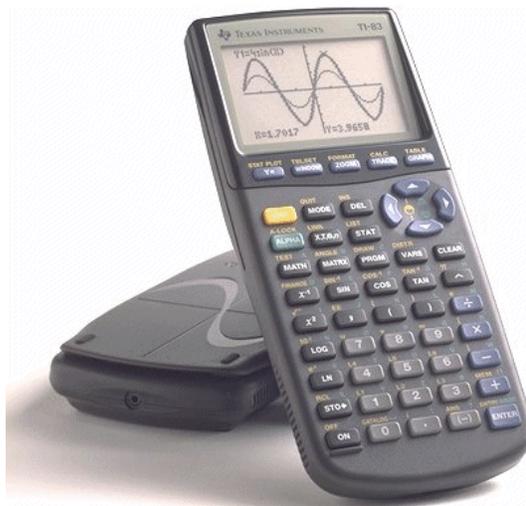
Solutions to Category 1
Mystery
Meet #5 - April, 2017

1) **Calculate the prime factorization of 1365:**
   1365 = 3 x 5 x 7 x 13.
   The average of the four factors is (3 + 5 + 7 + 13) / 4
   = 28 / 4 = 7.

2) **This problem can be solved algebraically or by using**
   a sense of proportion and equivalent ratios (fractions).
   Using algebra, let X = the current age, then X - 6
   = the age six years ago.

   $$\frac{x}{x-6} = \frac{5}{3}$$ **The data can be represented by this proportion.**

   5(x-6) = 3x   Cross products are equal.
   5x - 30 = 3x   Distribute.
   2x = 30   Rearrange terms.
   x = 15   Divide both sides by 2.

   The current age, x, is 15, so the age 17 years from now is 15 + 17, or 32.

3) **Ten to the 40th power looks like the digit 1 followed by 40 zeroes**
   (a 41-digit number). Subtracting 57 creates a 40-digit number, leaving 43
   in the tens and units places. All the rest of the remaining 38 digits are 9s.
   So, the sum of the 40 digits is (38)(9) + 43 = 342 + 4 + 3 = 349.

Answers
1)   7
2)   32
3)   349

1) A cube has a volume of 1331 cubic yards. How many square yards are in the total surface area?

2) Eight baseballs are packed tightly into a cubical box whole length is six inches. Each baseball is three inches in diameter. What percent of the box is empty space? Round your answer to the nearest whole percent. Use $\pi \approx 3.14.$

3) This plane figure, drawn on the x-y Cartesian coordinate plane, is rotated 180 degrees, using the y-axis as the axis of rotation. What is the volume, in cubic units, of the resulting 3-D object? The curve above the x-axis is a semi-circle. Use $\pi \approx 3.14.$ Round your answer to the nearest ten cubic units.
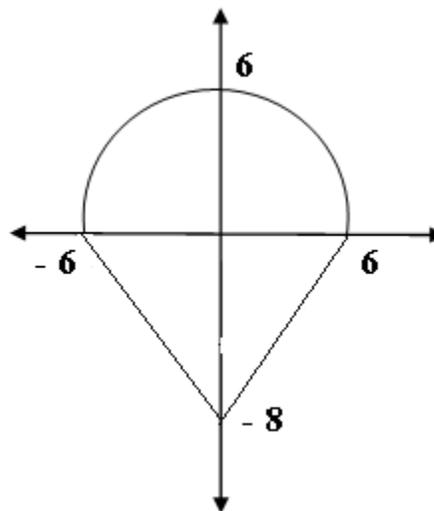
Answers
1) _____
2) _____ %
3) _____

Solutions to Category 2
Geometry
Meet #5 - April, 2017

1) The cube root of 1331, or 11, is the length of one side
    of the cube. So, the total surface area is (6)(11)(11)
    = 726 square yards.

2) The empty space = (the volume of the cube) - the total
    volume of the eight baseballs

$$= \left(\text{one side}\right)^3 - (8)\left(\frac{4}{3}\pi r^3\right)$$

$$= (6)^3 - (8)\left(\frac{4}{3}(3.14)(1.5)^3\right)$$

$= 216 - (8)((1.333)(3.14)(3.375))$
$= 216 - (8)(14.1265)$
$= 216 - (113.0.12)$
$= 102.988$

As a percent of the cube, the empty space is 102.988 / 216 = 0.4767 . . .
or about 48% when rounded to the nearest whole percent.

3) When rotated 180 degrees, the plane figure looks like an ice cream cone with
    a hemisphere on top and a cone on the bottom. The volume is given by this
    formula:

$$\text{Volume} = \left(\frac{1}{2}\right)\left(\frac{4}{3}\pi r^3\right) + \frac{1}{3}\left(\pi r^2 h\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{4}{3}\pi (6)^3\right) + \frac{1}{3}\left(\pi (6)^2 (8)\right)$$

$$= \left(\frac{1}{2}\right)\left((1.333)(3.14)(216)\right) + \frac{1}{3}\left((3.14)(6)^2 (8)\right)$$

$= 452.0469... + 301.409...$

$= 753.4559...$ or about 750 cubic units when rounded to
the nearest ten cubic units.

1) Set A = {multiples of 4 between 1 and 170}
    Set B = {multiples of 6 between 1 and 170}

    How many numbers are in the intersection of sets A and B ?

2) A group of 74 people saw either the movie "Waterfront Nightmare"
    or the movie "The Secret Village" or both. Fifty-six people saw the
    first movie while 39 saw the second. How many people saw both ?

3) Set C = { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }
    Set D = { 2, 3, 5, 7, 11, 13 }

    Set E = { common fractions $\dfrac{X}{Y}$ where X is an element of Set C
    and Y is an element of Set D }. How many elements are in Set E ?
    (Reminder: In a common fraction, the numerator and denominator
    do not share any common factors. The fraction may be proper or
    improper.

Answers

1) _____

2) _____

3) _____
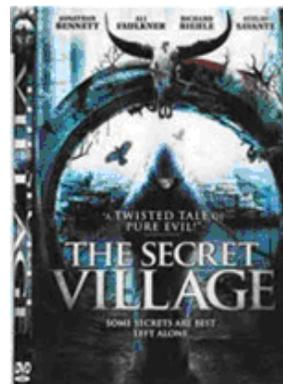
Solutions to Category 3
Number Theory
Meet #5 - April, 2017

1) The common multiple of 4 and 6 that lie between
    1 and 170 are multiple of 12, or
    { 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144,
      156, 168 }. There are 14 such multiples.

2) Let X = the number of people who saw both movies.
    So,   (56 + 39) - X = 74
                 95 - X = 74
                     X = 21
    Therefore, 21 people saw both movies.

3) The common (simplified) fractions are the (11)(6) possible fraction, minus
    the numbers from Set C that are multiple of 2 (there are six), minus the
    remaining multiples of 3 (there are two), minus the remaining multiples of
    5 (there is one), minus the remaining multiple of 7 (there is one), minus the
    remaining multiples of 11 (there is one).

        (11)(6) - (6 + 4 + 2 + 1 + 1)
    =   66 - 14
    =   52.

    Here is a full accounting of all the common fractions:

        numerators of 2:   2/3, 2/5, 2/7, 2/11, 2/13 (five)
        numerators of 3:   3/2, 3/5, 3/7, 3/11, 3/13 (five)
        numerators of 4:   4/3, 4/5, 4/7, 4/11, 4/13 (five)
        numerators of 5:   5/2, 5/3, 5/7, 5/11, 5/13 (five)
        numerators of 6:   6/5, 6/7, 6/11, 6/13   (four)
        numerators of 7:   7/2, 7/3, 7/5, 7/11, 7/13 (five)
        numerators of 8:   8/3, 8/5, 8/7, 8/11, 8/13 (five)
        numerators of 9:   9/2, 9/5, 9/7, 9/11, 9/13 (five)
        numerators of 10: 10/3, 10/7, 10/11, 10/13 (four)
        numerators of 11: 11/2, 11/3, 11/5, 11/7, 1/13 (five)
        numerators of 12: 12/5, 12/7, 12/11, 12/13 (four)

    There are 52 common fractions.

Category 4
Arithmetic   *Calculator meet*
Meet #5 - April, 2017

1) Caitlin has a box containing 17 forks, 24 teaspoons, and 19
    tablespoons. If she chooses a utensil at random, what is the probability
    that she chooses a teaspoon? Express your answer as a common
    fraction.

2) Twelve points are placed on the circumference of a circle. Three points
    are chosen randomly and connected to form a triangle. What is the
    maximum number of triangles that can be drawn?

3) Frank has a bag of fruit chews containing the following flavors:
    6 raspberry, 4 cherry, and 5 melon. If he chooses two at random,
    then what is the probability that they are the same flavor? Express
    your answer as a common fraction.

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 4
Arithmetic
Meet #5 - April, 2017

1) The total number of utensils is $17 + 24 + 19 = 60$.
   The probability of selecting a teaspoon is
   $$\frac{24}{60} = \frac{12}{30} = \frac{6}{15} = \frac{2}{5}$$

2) $12C3 = 220$.

3) The probability of choosing two of the same color
   = (probability of two raspberry) + (probability of two cherry) + (probability of two melon)
   $$= \left(\frac{6}{15} \times \frac{5}{14}\right) + \left(\frac{4}{15} \times \frac{3}{14}\right) + \left(\frac{5}{15} \times \frac{4}{14}\right)$$
   $$= \left(\frac{30}{210}\right) + \left(\frac{12}{210}\right) + \left(\frac{20}{210}\right)$$
   $$= \frac{62}{210} = \frac{31}{105}$$

Calculator Meet

1) If $(X+7)(5X-2) = ax^2+bx+c$, then what is the value of $a+b+c$?

2) A quadratic equation is graphed so that the x-intercepts are $(-2,0)$ and $(6,0)$ while the y-intercept is $(0,-36)$. When $x=10$, then what is the value of y ?

- 2

6

- 36
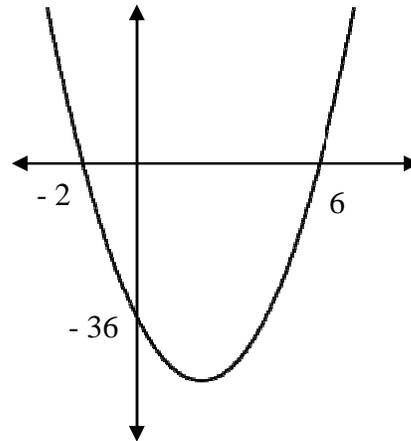
3) The quadratic equation $H=-4.9t^2+vt+h$ approximates the height in feet, H, that an object will attain after t seconds when launched vertically from a height of h meters with an initial upward velocity (starting velocity, or speed) of v meters per second. Once the object reaches its maximum height, gravity will draw the object back toward Earth. A candlepin bowling ball is tossed vertically into the air from a platform that is 58.8 meters above the water at an initial upward velocity of 19.6 meters per second.
How many seconds after the launch did it take for the bowling ball to hit the water ?

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 5
Algebra
Meet #5 - April, 2017

1) **Multiply the two binomials to determine the values of a, b, and c:**
   $(x+7)(5x-2) = 5x^2 + 35x - 2x - 14 = 5x^2 + 33x - 14$
   So, a = 5, b = 33, and c = - 14.
   a + b + c = 5 + 33 + (- 14) = 24

2) **First find the value of a (the coefficient of the first term) by using the data that the two x-intercepts yield the factors (x + 2) (x - 6) and then substituting the value of a third known point for x and y:**
   $$y = a(x+2)(x-6)$$
   $$-36 = a(0+2)(0-6)$$
   $$-36 = -12a$$
   $$a = 3$$

   Now substitute 10 for x: $y = 3(10+2)(10-6)$, **so** $y = 144$.

3) $H = -4.9t^2 + vt + h$ **the given formula**

   $H = -4.9t^2 + 19.6t + 58.8$ **Substitute 19.6 for v and 58.8 for h.**

   $0 = -4.9t^2 + 19.6t + 58.8$ **Set the equation to zero, the height of the water.**

   $0 = t^2 - 4t - 12$ **Divide both sides by - 4.9.**

   $0 = (t-6)(t+2)$ **Factor.**

   t = - 2 or t = 6   **Solve for t.**

   t = - 2 is extraneous, as it occurred prior to the launch, so we use t = 6.

   Therefore, it took 6 seconds for the bowling ball to hit the water.

Category 6
Team Round
Meet #5 - April, 2017

Each of the following nine problems is worth __four__ points.

1) Two lines, represented by the equations $2x + 3y = 11$ and $10x + My = 35$, always intersect on the x-y plane except when M is equal to what number ?

2) If $AB \neq 0$ and $4A = 0.4B$, then what is the value of $\dfrac{B}{A}$?

3) A parallelogram has vertices at $(0, 0)$, $(7, 0)$, and $(2, 4)$. What is the sum of the y-coordinates of all possible fourth vertices of the p-gram ?

4) In the figure at the right, $AC = 12$, $BC = 10$, and $AD = 9$. What is the value of BE ? Express your answer as a decimal.
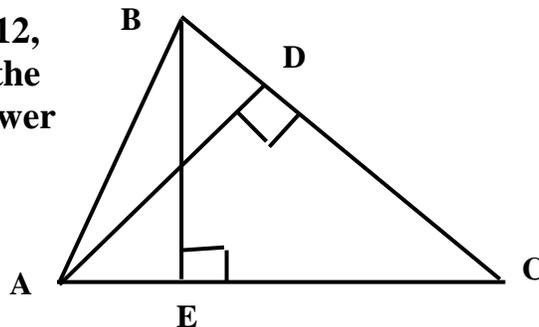
ANSWERS

1) _____

2) _____

3) _____

4) _____

5) _____

6) $ _____

7) _____

8) _____

9) _____

5) A regular hexagon is inscribed in a circle of radius 15 inches. How many square inches lie in the area between the circle and the hexagon ? Use $\pi \approx 3.142$. Round your final answer to the nearest tenth of a square inch.

6) An equal number of nickels, dimes, and quarters has a total value of $ 9.20. What is the total value of the quarters ?

7) After 3162 gallons of water were added to a large water tank that had already been filled to 75% of its capacity, the tank was then at 4/5 of its capacity. How many gallons of water does the tank hold when it is full ?

Problems #8 and #9 are on the next page.

8) **Four congruent isosceles triangles are attached to a square, as shown below. One side of the square = 48 cm. The distance from the midpoint of BC to A is 40 cm. The four triangles are folded along the edges of the square until their tips meet at point A. Find the number of cubic centimeters in the volume of the resulting shape.**

B

A

C
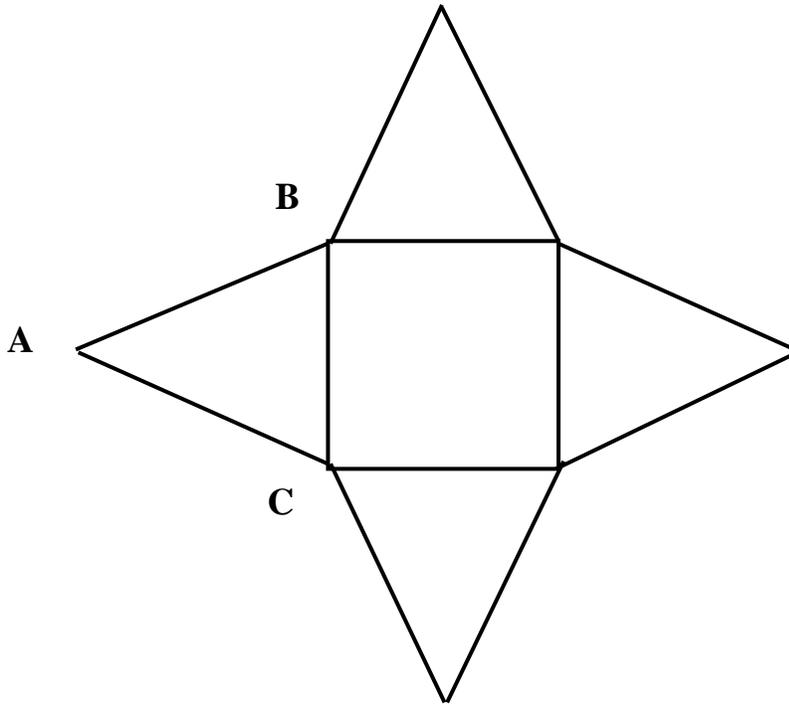
9) **Circle O has points A and B   on its circumference, as shown. OC = AC = AB. What is the value of x if it is the measure, in degrees, of angle AOC ?**

A

B

x

O

C
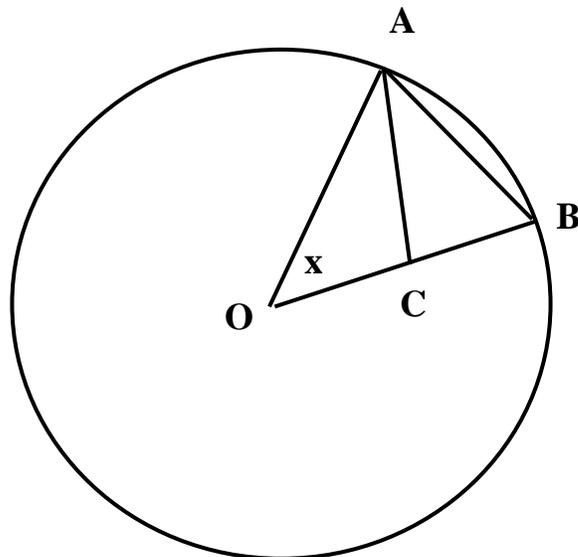
Solutions to Category 6
Team Round
Meet #5 - March, 2016

ANSWERS	
1)	15
2)	10
3)	4
4)	7.5
5)	122.4
6)	5.75
7)	63,240
8)	24,576
9)	36

1) For the two lines to not intersect, they must be parallel, so their slopes must be equal. The slope of the first line is - 2/3. For the second line to have that slope, - 10/M = - 2/3 . . . - 2M = - 30, so M = 15.

2) Scrambling the terms of the equation, B/A = 4/0.4 = 10.

3) The three possible fourth vertices are
$(- 5, 4)$, $(9, 4)$, and $(5, - 4)$. The sum of the y-coordinates of those three points is $4 + 4 + (- 4)$, or **4.**

4) There are two opportunities to multiply base x altitude in this triangle:
(1) (BC) x (AD) = (10 (9) = 90, and
(2) (AC) x (BE) . . . same triangle!
AC = 12, so (12)(BE) = 90, and BE = 7.5.

5) If radii are drawn from the center of the circle to the six vertices of the hexagon, then the hexagon can be seen as six congruent equilateral triangles. The area of the circle is $\pi r^2 \approx (3.142)(15)(15) \approx 706.95.$
The area of the hexagon is

$$(6)\left(\frac{s^2}{4}\sqrt{3} \right) \approx (6)\left[\frac{15^2}{4}(1.732) \right] \approx 584.55.$$

The area of the space between the hexagon and the circle is about 706.95 - 584.55, or **122.4.**
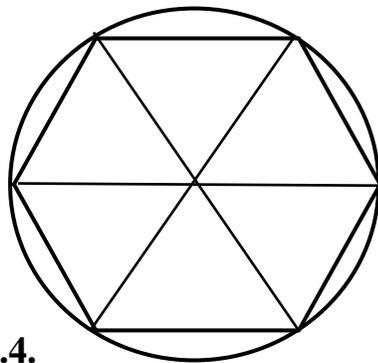
6) Let X = the number of each type of coin. This equation indicates that the total value of all the coins is $9.20:
$$5X + 10X + 25X = 920$$
$$40X = 920$$
$$X = 23$$
The value of the 23 quarters is (23)(0.25) = **$5.75**

7) The 3162 gallons occupy the space between 75% capacity and 4/5, or 80% capacity. So, 3162 is 5% of the capacity of the tank, and (3162)(20) is the full capacity of the tank = 63,240 gallons.

8) When the triangles are folded up, the resulting figure is a pyramid. The Pythagorean Theorem can be used to figure the altitude of the pyramid, or it can be recognized that the numbers are multiples of a known Pythagorean triple (3-4-5), scaled by a factor of 8, so the altitude of the pyramid is (8)(4) = 32.
The volume of the pyramid is　　　(1/3)(area of base)(altitude)
　　　　　　　　　　　　　　　= (1/3)(48)(48)(32)
　　　　　　　　　　　　　　　= 24,576 cubic centimeters.

9) Combine several relationships in order
   to resolve the value of x:
   (1) y = 2x, as the exterior angle
        of a triangle equals the sum
        of the two remote
        interior angles.
   (2) y = x + z, as the opposite
        angles of an isosceles
        triangle are equal
        (because the two radii,
        OA and OB, are equal.)
   (3) z = 180 - 2y, as the sum of
        the angles of a triangle is
        180 degrees.
   Solving: Substitute 180 - 2y for z
   from equation (3) into equation (2) to create y = x + 180 - 2y, or
   3y = x + 180   and   x = 3y - 180.
   Then substitute the result into equation (1) to create y = 2(3y - 180)
   . . .   y = 6y - 360 . . . 5y = 360 . . . y = 72. Then x = 36.
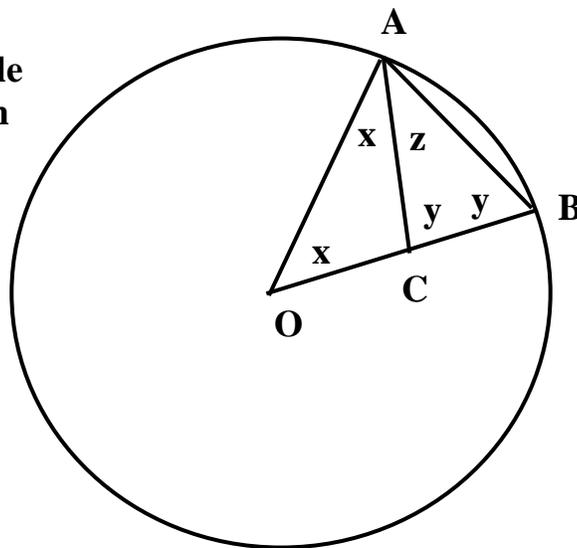
   This is an alternative solution, albeit a tough one! It may take many
   team members to put their heads together to come up with a strategy
   for a solution. The diagram alone does not provide adequate
   information. What is required is to draw auxiliary lines to create
   multiple relationships that will resolve the value of x. Extend the
   radius OB to create diameter BD. Then draw DA to create right
   angle DAB (an angle inscribed in a
   semicircle is a right triangle.
   For all three isosceles triangles,
   their base angles are equal.

The angles marked "z" are opposite sides OD and OA, both radii of triangle OAD. The angles marked "x" are opposite equal sides OC and AC. The angles marked "y" are opposite equal sides AB and AC. This equation, $z + (z + x + (180 - 2y) + y = 180$, comes from the idea that the three angles of triangle ABD have a sum of 180 degrees.

This equation, $x = z + z$, comes from the idea that an external angle of a triangle (OAD) is equal to the sum of the two remote interior angles. This equation, $z + x + (180 - 2y) = 90$, comes from the fact that angle DAB is a right angle.

(1) $z + (z + x + (180 - 2y) + y = 180$, so $2z + x - y = 0$, or $y = 2z + x$.

(2) $x = 2z$.

(3) $z + x + (180 - 2y) = 90$, then substitute from the first two equations to yield $z + 2z + (180 - 2(2z + 2z) = 90$, then $3z + (180 - 8z) = 90$, and $- 5z = - 90$, and $z = 18$. So, $x = 2z = (2)(18) = 36$ degrees.