

IMLEM Meet #3
January, 2016

Intermediate
Mathematics League
of
Eastern Massachusetts



**Solutions to Category 1
Mystery
Meet #3 - January, 2016**

1) The two numbers would be approximately at the square root of 552. Guessing and checking would yield 23 and 24 as the two numbers. Their sum is $23 + 24$, or 47.

2) The ratio of 18 to 26.4 can be scaled so that they have whole number values. Scaling by a factor of 10 gives a ratio of 180 to 264.

$$180 = 2 \times 2 \times 3 \times 3 \times 5$$

$$264 = 2 \times 2 \times 2 \times 3 \times 11$$

Their GCF is $2 \times 2 \times 3$, or 12.

Simplifying the ratio to lowest terms, by a factor of their GCF of 12 gives us a ratio of 15 to 22. The sum of $15 + 22 = 37$.

3) Using the principle of $\text{Rate} \times \text{Time} = \text{Distance}$, Dividing 360 meters by 120 seconds (2 minutes converted to seconds) yields 3 meters per second walking. At six times that rate, Marlene slid down the hill at 18 meters per second. Dividing the distance of 360 meters by the rate of 18 meters per second gives 20 seconds.

Answers

1) 47

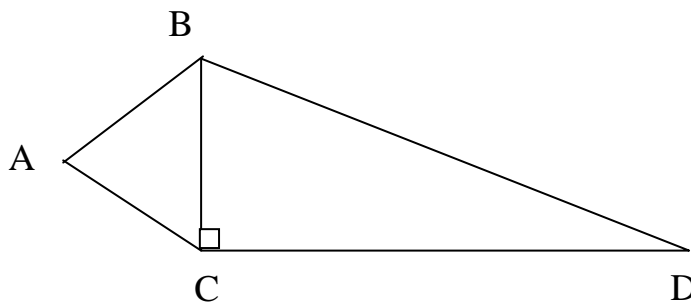
2) 37

3) 20

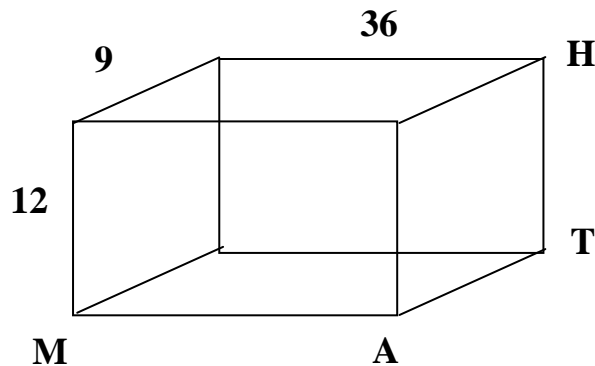
Category 2
Geometry
Meet #3 - January, 2016

1) All possible diagonals of a convex polygon are drawn. There are nine diagonals drawn from any one of the vertices. How many sides does this polygon have?

2) Triangle ABC is equilateral. $AB = 10$ inches. The area of right triangle BCD is 120 square inches. How many inches are in the perimeter of quadrilateral ABDC ?



3) An empty rectangular box has dimensions 12 cm by 9 cm by 36 cm as shown in the diagram (not drawn to scale). A bee trapped inside flew in a straight path from M to H. How many centimeters less did the bee fly than the spider that walked in straight paths from M to A to T to H ?

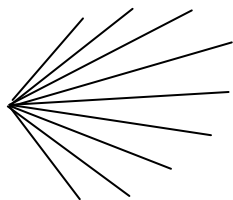


<u>Answers</u>	
1)	_____
2)	_____
3)	_____

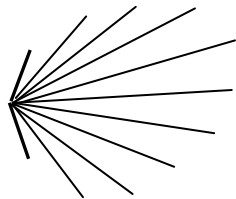
**Solutions to Category 2
Geometry
Meet #3 - January, 2016**

- 1) Students can draw the figure, as it is reasonably achievable. Another way is to analyze what happens at any given vertex. From one vertex are drawn nine diagonals and two sides of the polygon. The remote endpoints of the nine diagonals can be connected to form eight sides of the polygon. The remaining sides can then be drawn to make a total of 12 sides.

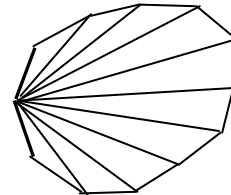
<u>Answers</u>	
1)	12
2)	70
3)	18



Here are the nine diagonals from one vertex.



Added are the two sides drawn from that vertex (bold).



Here is the 12-gon with all 12 sides drawn.

- 2) Since triangle ABC is equilateral, $BC = 10$. The area of triangle BCD is 120 square inches, so $\frac{1}{2}BH = \frac{1}{2}(CD)(10) = 120$, then $CD = 24$. Using the Pythagorean theorem to find the length of BD:

$$\begin{aligned} (BC)^2 + (CD)^2 &= (BD)^2 \\ (10)^2 + (24)^2 &= (BD)^2 \\ 100 + 576 &= (BD)^2 \\ 676 &= (BD)^2 \\ 26 &= BD \end{aligned}$$

The perimeter of ABDC = $10 + 10 + 24 + 26 = 70$ inches.

- 3) The space diagonal, MH, can be found by applying the Pythagorean theorem twice - once to find the diagonal of the floor and a second time to find MH . . . or all the calculations can be combined into one expression:

$\sqrt{9^2 + 12^2 + 36^2} = \sqrt{81 + 144 + 1296} = \sqrt{1521} = 39$. This is how many cm the bee flew. The spider walked $MA + AT + TH = 36 + 9 + 12 = 57$ cm, so the bee flew $57 - 39$, or 18 cm less than the spider walked.

Category 3
Number Theory
Meet #3 - January, 2016

1) 10110 is a base three numeral. What is its value in base 10 ?

2) Express the answer to the calculation below in scientific notation:

$$\frac{1.5 \times 10^{-4}}{75 \times 10^2}$$

3) The decimal (base 10) numeral 427.608 means

$$(4 \times 10^2) + (2 \times 10^1) + (7 \times 10^0) + (6 \times 10^{-1}) + (0 \times 10^{-2}) + (8 \times 10^{-3}).$$

The base 4 numeral 213.23 means

$$(2 \times 4^2) + (1 \times 4^1) + (3 \times 4^0) + (2 \times 4^{-1}) + (3 \times 4^{-2}).$$

For each numeral, the "point" separates the whole number part from the fractional part.

Express the base 8 numeral 53.21 as a base 2 numeral, including the point.

Answers

1) _____

2) _____

3) _____

**Solutions to Category 3
Number Theory
Meet #3 - January, 2016**

1) 10110 (base 3) = $81 + 9 + 3 = 93$.

2) $\frac{1.5 \times 10^{-4}}{75 \times 10^2} = \frac{150 \times 10^{-2} \times 10^{-4}}{75 \times 10^2} = 2 \times 10^{-2} \times 10^{-2} \times 10^{-4} = 2 \times 10^{-8}$

<u>Answers</u>	
1)	93
2)	2×10^{-8}
3)	101011.010001

3) First expand and evaluate the base 8 numeral 53.21 from left to right):
 = $5(8) + 3(1) + 2(1/8) + 1(1/64)$
 = $40 + 3 + 1/4 + 1/64$
 = 43 and $17/64$

Since the denominator, 64, is a power of 2, we must express $17/64$ as the sum of fractions whose denominators are powers of 2 and whose numerators are each less than 2. The fraction $1/4$ uses up $16/64$, leaving just $1/64$ remaining. The whole number part, 43, can be expressed as the sum of these powers of 2: $32 + 8 + 2 + 1$. The answer can be computed as

$$(1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) +$$

$$(0 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3}) + (0 \times 2^{-4}) + (0 \times 2^{-5}) + (1 \times 2^{-6})$$

= **101011.010001**

Category 4
Arithmetic
Meet #3 - January, 2016



1) Find the value of $6^0 + 5^1 + 4^2 + 3^3 + 2^4 + 1^5$.

2) Evaluate. Express your answer as a decimal: $\sqrt[3]{\frac{2^{12}}{(0.1)^{-6}}}$

3) Express the answer to the following as a decimal:

$$8^{-1} + 2^{-4} - 5^{-2} - 4^{-3}$$

ANSWERS

1) _____

2) _____

3) _____

**Solutions to Category 4
Arithmetic
Meet #3 - January, 2016**

Answers

1) 66

2) 0.16

3) 0.131875

1) $6^0 + 5^1 + 4^2 + 3^3 + 2^4 + 1^5 = 1 + 5 + 16 + 27 + 16 + 1 = 66.$

2)

$$\sqrt[3]{\frac{2^{12}}{(0.1)^{-6}}} = \sqrt[3]{\frac{2^{12}}{\left(\frac{1}{10}\right)^{-6}}} = \sqrt[3]{2^{12} \times \left(\frac{1}{10}\right)^6} = 2^{12 \times \frac{1}{3}} \times \left(\frac{1}{10}\right)^{6 \times \frac{1}{3}} = 2^4 \times \left(\frac{1}{10}\right)^2 = 16 \times 0.01 = 0.16$$

3) $8^{-1} + 2^{-4} - 5^{-2} - 4^{-3} = \frac{1}{8} + \frac{1}{16} - \frac{1}{25} - \frac{1}{64}$

$$= 0.125 + 0.0625 - 0.04 - 0.015625$$

$$= 0.131875$$

Category 5

Algebra

Meet #3 - January, 2016

- 1) What are the two values of N that make this absolute value equation true?

$$|N - 7| = 12$$

- 2) The graph represents the solution set of the inequality

$$-3|X + 6| - 17 \geq -86$$



What is the value of $A + B$?

- 3) Solve the following inequality for C :

$$5(C + 2) - 4(2C - 3) + 7(3C - 8) < -6(0.5C - 9) - 64$$

Express your answer as a common fraction.

ANSWERS

1) ___ and ___

2) $A + B =$ ____

3) $C <$ _____

Solutions to Category 5
Arithmetic
Meet #3 - January, 2016

Answers

1) **- 5 ; 19**
(any order)

2) **- 12**

3) **$\frac{8}{7}$**

1) Either $N - 7 = 12$ or $N - 7 = -12$. So, either $N = 19$ or $X = -5$.

2) $-3|X+6| - 17 \geq -86$ original inequality
 $-3|X+6| \geq -69$ having added 17 to both members
 $|X+6| \leq 23$ having divided both members by -3

$$X + 6 \leq 23 \text{ and } X + 6 \geq -23$$

$$X \leq 17 \text{ and } X \geq -29$$

$$\text{So, } A + B = -29 + 17 = -12$$

3) $5(C + 2) - 4(2C - 3) + 7(3C - 8) < -6(0.5C - 9) - 64$ original inequality
 $5C + 10 - 8C + 12 + 21C - 56 < -3C + 54 - 64$ distribute
 $18C - 34 < -3C - 10$ combine terms
 $21C < 24$ add 3C; add 34
 $C < 8/7$ divide 21

Category 6
Team Round
Meet #3 - January, 2016

Each of the following nine problems is worth four points.

- 1) If $\frac{1}{3} + \frac{5}{6} = \frac{21}{N}$, then what is the value of N ?
- 2) The reciprocal of 8 is how much more than the reciprocal of 20 ?
Express your answer as a decimal.
- 3) If the difference between the total number of diagonals in a convex polygon and the number of diagonals that can be drawn from any one of its vertices is 91, then how many sides does the polygon have?
- 4) There are 63 ants at a picnic. Thirty-four are red, 20 are poisonous, and 15 are neither red nor poisonous. In lowest terms, the ratio of the number of poisonous red ants to the number that are neither poisonous nor red is C:D. What is the value of the product CD ?

ANSWERS

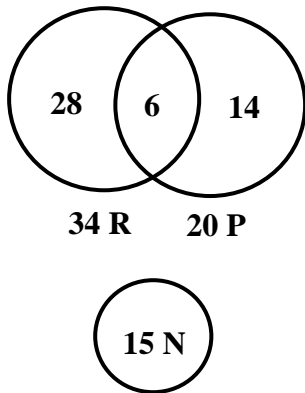
- 1) _____
- 2) _____
- 3) _____
- 4) _____
- 5) _____
- 6) _____
- 7) _____
- 8) _____
- 9) _____

- 5) The perimeter of a rectangle is 324 feet. What is its greatest possible area in square inches?
- 6) What month occurs 10,000 days after January 1 ?
- 7) If 5 pots = 4 vases, and 3 boxes = 4 pots, then how many vases = 15 boxes?
- 8) How many two-digit positive integers are twice the sum of their digits?
- 9) The sides of a regular 15-sided convex polygon are numbered counter-clockwise and consecutively from 1 through 15. The polygon is rotated clockwise 3408 degrees. What side is in the position that side 2 occupied before the rotation?

**Solutions to Category 6
Team Round
Meet #3 - January, 2016**

ANSWERS

- 1) 18
- 2) 0.075
- 3) 16
- 4) 10
- 5) 944,784
- 6) May
- 7) 16
- 8) 1 (or "one")
- 9) 9



- 1) $1/3 + 5/6 = 2/6 + 5/6 = 7/6$. Scale by a factor of 3 to give $21/18$. So, $N = 18$.
- 2) The reciprocal of 8 is $1/8 = 0.125$.
The reciprocal of 20 is $1/20 = 0.05$.
 $0.125 - 0.05 = 0.075$.
- 3) The chart below is one way to organize data.
Let S = the number of sides, D = the number of diagonals from one vertex, T = the total number of diagonals in the polygon and d = the difference between T and D .

<u>S</u>	<u>D</u>	<u>T</u>	<u>d</u>
3	0	0	0
4	1	2	1
5	2	5	3
6	3	9	6
7	4	14	10
8	5	20	15
9	6	27	21
10	7	35	28
11	8	44	36
12	9	54	45
13	10	65	55
14	15	77	62
15	12	90	78
16	13	104	91

So, the number of sides is 16.

- 4) A Venn diagram, like the one above, displays the relationship among the quantities. Of the 63 ants, the 15 that are neither red nor poisonous are in the bottom circle. The leaves 63 - 15, or 48 ants that are either red or poisonous or both, all represented in the interlocking

(NEXT PAGE FOR THE REST OF THE SOLUTIONS)

circles. The number of red ants plus the number of poisonous ants exceeds the 48 by 6, so 6 are both red and poisonous (the shared space). So, the ratio of the number of poisonous red ants (6) to the number that are neither red nor poisonous (15) is $C:D = 6:15$. In lowest terms, this is $2:5$. $CD = (2)(5) = 10$.

- 5) The maximum area of a rectangle with a given perimeter occurs when it is a square. Divide 324 by 4 to get the length of one side of that square = 81 feet. The area is $(81)(81) = 6561$ square feet. The question asks for the area in square inches, so multiply 6561 square feet by 144 square inches per square foot to get 944,784 square inches.
- 6) Divide 10,000 days by 365 days per year to get the approximate number of years, which is about 27.4 years. Then $27 \times 365 = 9855$. So, 27 years is about 9855 days (about 145 shy of 10,000) which brings us to about another January 1. Adding the approximately 145 days sends us into May. The few added days due to leap years, over a period of 27 years, would not be enough to alter the idea that 10,000 days after January 1 sends us somewhere into the month of May.
- 7) If 3 boxes = 4 pots, then scale up by a factor of five to get 15 boxes = 20 pots. Then scale up 5 pots = 4 vases by a factor of four to get 20 pots = 16 vases. Now 20 pots is the common link:
If 15 boxes = 20 pots and 20 pots = 16 vases, then 15 boxes = 16 vases.
- 8) Let U = the units digit and T = the tens digit.
Then $10T + U = 2(T + U)$. Normally, we seek another relationship in the problem to get another equation with these two variables. However, that won't be necessary in this case: $10T + U = 2T + 2U$, then $8T = U$. Now consider that only whole number values of T and U solve this equation. So, $U = 8$ and $T = 1$, giving us the number 18. Check: 18 is twice the sum of its digits. Hence, there is only one such number: 18.

(NEXT PAGE FOR THE FINAL SOLUTION)

9) Divide 3408 by 360 to find the number of complete rotations the polygon made (9.4666...) so, nine rotations and then some. Multiply 9 by 360 to get 3240, which is the number of degrees the polygon rotates in 9 complete turns. $3408 - 3240 = 168$, the number of extra degrees the polygon rotates beyond its original position. A regular 15-gon has 15 equal angles of 24 degrees each. $168 / 24 = 7$, which is the number of sides counted in the counter-clockwise rotation beyond the polygon's original position, thus replacing side #2 with side #9. *** Note that the sides were originally numbered in a COUNTER-CLOCKWISE manner but then the polygon was rotated CLOCKWISE.