Intermediate Mathematics League of Eastern Massachusetts

Statistics and notes – not part of the original meet

Scheduled Meet Date	March 29, 2012	
Number of Teams Competing	72	
Average Team Score	82	
Average Individual Score	7.2	(out of 18)

Category	1 Myst	2 Geom	3 NumTh	4 Arith	5 Alg
Number of Regulars Competing in This Category	414	408	407	405	399
Percent of Regulars with each possible score in the category:					
0	29%	34%	19%	**46%**	19%
2	**36%**	**38%**	**39%**	32%	25%
4	25%	15%	35%	18%	**29%**
6	11%	14%	7%	3%	28%

Team Round: Mean 13.9. # Team Round Perfect Scores of 36: 9 teams.

Category 1 – Mystery

1. Ed, Fred, Ned, and Ted are at the playground. When they stop to rest, there are two different benches they can sit on. How many different ways do they have to arrange themselves on the benches?
   (What we care about is who sits on what bench, not the order of people on each bench).

2. If I add 6 inches to one side of a square, and 4 inches to the other side, I get a rectangle whose area is twice as much as the original square's area.
   How many inches were in the original square's side?

3. Eight swimmers are competing in a race. How many different possibilities are there for the 3 medalists? (The winners of the gold, silver, and bronze medals).
   Assume there are no ties.

Answers

1. ________________

2. ____________inches

3. ________________

Solutions to Category 1 – Mystery

Answers
1. 16
2. 12
3. 336

1. Since each person has 2 choices as to where to sit, the total number of possibilities is $2^4 = 16$.
   This can be thought of also as the number of subsets that the group has (we choose a subset to be on one bench, and the rest of them go on the other bench, thus defining the arrnagement).

2. If we call the length of original square's side L, then the information we're given can be written as:
   $(L+6) * (L+4) = 2 \cdot L^2$ which we can simplify to
   $-L^2 + 10 \cdot L + 24 = 0$ and solve to get $L = 12$ inches (the other solution $L = -2$ obviously cannot serve us).

3. The gold medalist can be anyone of the eights swimmers, the silver medalist can be anyone of the remaining seven swimmers, and the bronze medalist can be anyone of the remaining six. Overall we have $8 \cdot 7 \cdot 6 = 336$ different variations for the three medalists.

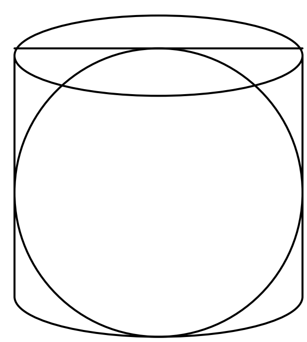
Category 2 – Geometry

Use $\pi = 3.14$ whenever necessary

1. How many cubes that are $1'' \times 1'' \times 1''$ have a combined surface area as a cube that is $8'' \times 8'' \times 8''$?

2. How many spherical raindrops with a diameter of 2 millimeters will it take to fill a cylindrical bucket whose radius is 314 millimeters and height is 400 millimeters?

3. A cylinder is built around a ball (see diagram below) so that they have the same radius and that the cylinder's height equals the ball's diameter.
   If the ball's volume is 200 cubic centimeters (*cc*), then how many *cc* are there in the cylinder's volume?



Answers

1. ________________

2. ________________

3. _____________*cc*

Solutions to Category 2 – Geometery

Answers

1. 64
2. 29,578,800
3. 300

1. A cube has 6 faces, so its surface area is 6 times the area of a face. The little cube's face has an area of 1 suare inch, and the large cube has a face with an area of 64 square inch, so it will take 64 little cubes to match the surface area of the larger one (and, of course, 8^3 cubes to match the volume).

2. The volume of a raindrop is $\frac{4}{3} \cdot \pi \cdot 1^3$ cubic millimeter.

   (The radius of the drop is 1 millimeter)

   The volume of the bucket is $\pi \cdot 314^2 \cdot 400$ cubic millimeter and so it will take:

   $\frac{\pi \cdot 314^2 \cdot 400}{\frac{4}{3} \cdot \pi} = \frac{3 \cdot 314^2 \cdot 400}{4} = 29{,}578{,}800$ raindrops to fill.

3. A ball's volume is given by $V_{ball} = \frac{4}{3} \cdot \pi \cdot R^3$ and a cylinder's volume is given by $V_{cylinder} = \pi \cdot R^2 \cdot H$.

   In our case the radii are the same, and $H = 2 \cdot R$ and so $V_{cylinder} = 2 \cdot \pi \cdot R^3$ or in other words $V_{cylinder} = \frac{3}{2} \cdot V_{ball} = 300\ cc$

Category 3 – Number Theory

1. In a class of 30 students, 12 students signed up for piano lessons and 17 students signed up for guitar lessons. If 4 students signed up for both kinds of lessons, then how many students didn't sign up for either one?

2. From Maine to Florida, there are 15 states on the East Coast (Pennsylvania included here). You want to plan a trip along the east coast that will visit at least 4 of these states. How many possible lists of states visited can you have? *(The order in which you visit them does not matter.)*

3. You have 100 photographs on your computer and use a program that lets you label photographs with a tags (labels). You use 3 different tags ('Travel', 'Friends', 'School'), and each photo can have any or none of the tags. Overall, you've used tags 240 times. What is the smallest possible number of photographs that have all three tags?

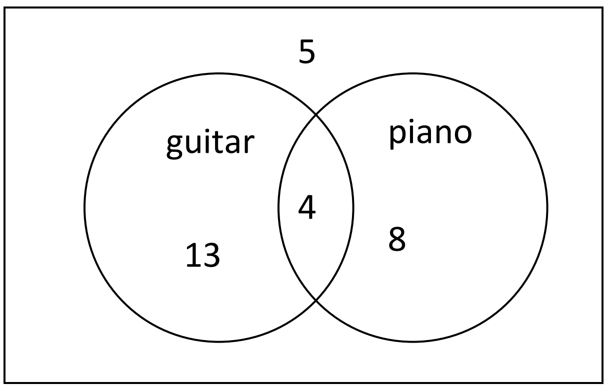
Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3 – Number Theory

Answers

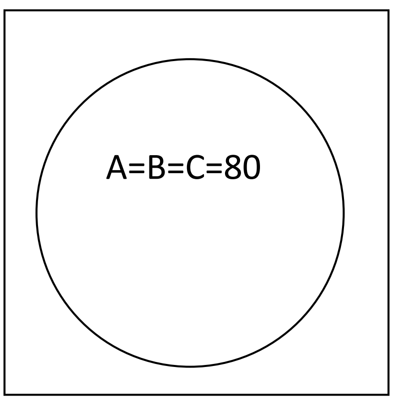
1. 5
2. 32,192
3. 40

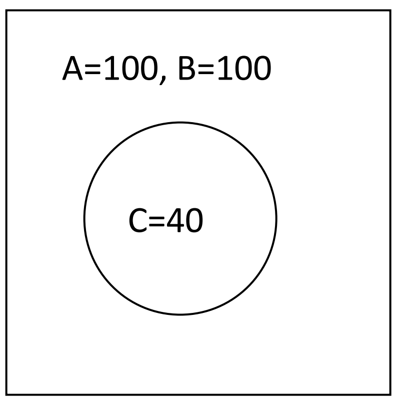
1. Given that the intersection of the two sets (piano and guitar) is of size 4, we know there are 8 piano-only students, and 13 guitar-only students. Overall, 25 students take lessons on an instrument $(17 + 12 - 4)$, which leaves 5 students who do not.

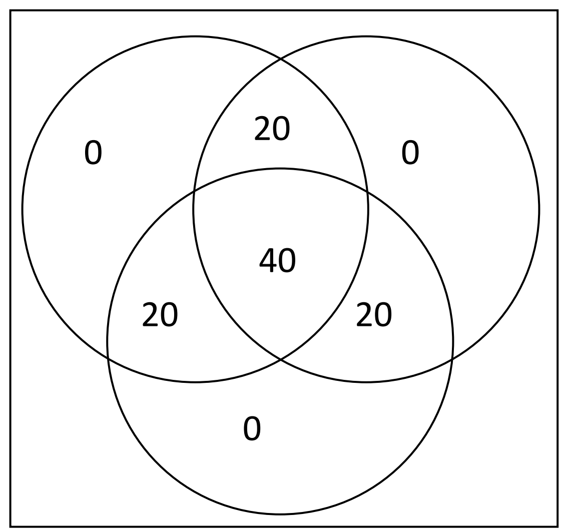


2. For the set of 15 states, there are 2^{15} possible subsets. One of these is the empty set, 15 of these include one state only, ${}_2C_{15}$=105 include only two states, and ${}_3C_{15}$=455 include only three states. So the answer is:
$2^{15} - 1 - 15 - 105 - 455 = 32{,}192$ possible lists.

3. In one extreme distribution (left diagram), all three tags are used on the same 80 pictures (leaving 20 images untagged), with an overlap of 80 (clearly that is the greatest overlap possible, as we can't increase the size of all groups at the same time).
   On the other extreme (middle diagram), we can have one tag used only 40 times, while using each of the other two tags 100 times, resulting in an overlap of 40. Can we reduce the size of this overlap? No – since in order to do that we have to remove a tag from one of these 40 images, but we have to use that tag elsewhere (to keep the total at 240) but the only place to use a tag is on images already tagged twice, thus maintaining the number of triple-tagged images at 40. The diagram on the right shows a case where $A = B = C = 80$ and the overlap is 40.







Algebraically, if we call the number of triple-tagged photos x, then we use $3 \cdot x$ tags on them, and can use as many as $2 \cdot (100 - x)$ tags on the rest of the photos (tagging each other image twice). Making the sum equal 240 results in $x = 40$

Category 4 – Arithmetic

1. A restaurant lets children build their own ice-cream sundae:
   They can select 2 scoops of ice-cream from a variety of 4 flavors, they can decide whether to add syrup and/or whipped cream, and have the whole thing served in a cup or a cone. How many different sundaes can be made?
   Note: Your two scoops may be of the same flavor. You cannot select only one scoop. Assume that the order of the scoops or toppings does not matter, so chocolate-vanilla would be the same as vanilla-chocolate.

2. Rolling a fair six-sided die 6 times, what is the probability you never get a '6'?
   Express you answer as a percent, rounded to the nearest whole percentage.

3. A box contains 4 balls of each color: Red, Green, Yellow, and Blue (for a total of 16 balls overall). If two balls are selected at random, what is the probability that they are of the same color?
   Express your answer as a simple fraction.
   To be clear, the two balls are selected without replacement: one is taken out, then a second.

Answers

1. ________________

2. ______________ %

3. ________________

Solutions to Category 4 - Arithmetic

Answers
1. 80
2. 33%
3. $\frac{1}{5}$

1. All the choices are independent of each other, so we multiply the number of options for each decision:
   Cone or cup? = 2 options. Yes or no syrup? = 2 options
   Yes or no whipped cream? = 2 options
   Selecting 2 flavors out of 4 = we can have 4 options if the scoops are identical, and 6 (${}_2C_4$) more if they're not, for a total of 10 options.
   Overall we get $10 \cdot 2 \cdot 2 \cdot 2 = 80$ possible sundaes.
   Editor note: The original question did not mention scoop order. Student Ben Schiffer argues that the order of the scoops matters, which would give an answer of 128, or that the question should be clarified, which was done here.
   Another person suggested: The wording didn't mention order, so simpler interpretation is preferred. This argument could be extended to the order of the whole sundae: What if we want the whipped cream on bottom and the syrup in between the two scoops? Is that a different sundae?

2. With a fair die, each result has a $\frac{1}{6}$ probability at each throw. We therefore have a $\frac{5}{6}$ probability at each throw not to get a '6', and in six consecutive rolls this becomes $(\frac{5}{6})^6 = \frac{15625}{46656} = 0.33489 \ldots \cong 33\%$
   (We multiply the probabilities as the rolls are independent of each other).

3. As with most combinatorical problems we can arrive at the solution in many different ways. One way is to ask how many ways to pick 2 Blue balls out of 4 blue balls? In this case, the answer is ${}_2C_4$=6, and similarly there are 6 ways to pick two balls of the same color for the other 3 colors, for a total of 24 ways of selecting two of the same color. Overall, there are ${}_2C_{16}$=120 ways of picking 2 balls out of the box, so the required pobability is $\frac{24}{120} = \frac{1}{5}$
   Another way to think about it: It doesn't matter which ball we pick first, but after we do, we have 15 choices for the 2nd ball, and three of those are balls of the same color as the first one, so again $\frac{3}{15} = \frac{1}{5}$ probability of that.

Category 5 – Algebra

1. A rectangle is 3 inch longer than it is wide. Its area is 550 square inches. How many inches are there in its perimeter?

2. The price of a square solar panel is calculated in the following way:
10 cents for each square inch in its area, plus 2.5 cents for each inch in its perimeter.
How many inches are there in the side of a solar panel that costs $3?

3. There are two solutions to the quadratic equation $x^2 + B \cdot x + C = 0$.
Their sum is 7 and the positive difference between them is 5.
What is the value of the parameter C?

Answers
1. ____________inches
2. ____________inches
3. ________________

Solutions to Category 5 - Algebra

Answers

1. 94
2. 5
3. 6

1. If we call the width x, then we know that $x \cdot (x+3) = 550$ or $x^2 + 3x - 550 = 0$ which we can solve:
   $x = \frac{-3 \pm \sqrt{9+2200}}{2} = \frac{-3 \pm 47}{2}$ and the positive solution is $x = 22\ inch$
   Therefore the perimeter is $2 \cdot (22 + 25) = 94\ inches$

2. If we call the length of a side x, then we know that:
   $10 \cdot x^2 + 2.5 \cdot 4 \cdot x = 300$ (note we have to translate the dollars to cents).
   Solving this we get: $x_{1,2} = \frac{-10 \pm \sqrt{10^2 + 4 \cdot 10 \cdot 300}}{20} = \frac{-10 \pm 110}{20}$
   and the positive solution is $x = 5$ inches.

3. Given that the solutions of any quadratic equation are given by $x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4 \cdot C}}{2}$ ($A = 1$ in our case), we get that their sum is $x_1 + x_2 = -B$ and their difference is $x_1 - x_2 = \sqrt{B^2 - 4 \cdot C}$. So we're told that $-B = 7$ and also that $\sqrt{B^2 - 4 \cdot C} = \sqrt{49 - 4 \cdot C} = 5$ and we conclude that $4 \cdot C = 24$ or $C = 6$

Category 6

1. Danny's balloon takes on a spherical shape when filled with water. Danny filled it first with 5 liters of water. How many **more** liters does he need to fill in order to quadruple the balloon's surface area? *Quadruple ≡ Multiply by 4*

2. Sarah owns 10 hardcover books, 10 paperback books, and 10 books on her electronic reader. There are three titles she owns in all three formats, and three others she has in two formats. How many different titles does she own overall?
Note: Within each format, all ten titles are different from each other.

3. How many different ways are there to organize 6 books on two shelves such that there are exactly 3 books on each shelf (and the order of books on each shelf matters)?

4. What is the sum of the *natural* solutions to: $x^4 - 13 \cdot x^2 + 36 = 0$?

5. An airline serves five cities on the East coast (call them $E1, E2, E3, E4, E5$) and five on the West coast ($W1, W2, W3, W4, W5$). There are direct flights between any two cities. Starting at city $E1$, how many different trips can you make which are made up of 3 flights if you have to visit at least one city on the West coast on the trip? *(That is the only requirement).*
(An example of such a trip is $E1 \rightarrow W1 \rightarrow W2 \rightarrow E3$)

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$A - B - \frac{C - E}{D}$$

Answers

1. ____________=*A*
2. ____________=*B*
3. ____________=*C*
4. ____________=*D*
5. ____________=*E*
6. ______________

Solutions to Category 6

Answers
1. 35
2. 21
3. 720
4. 5
5. 665
6. 3

1. Let's call the initial radius (after putting in 5 liters) R.

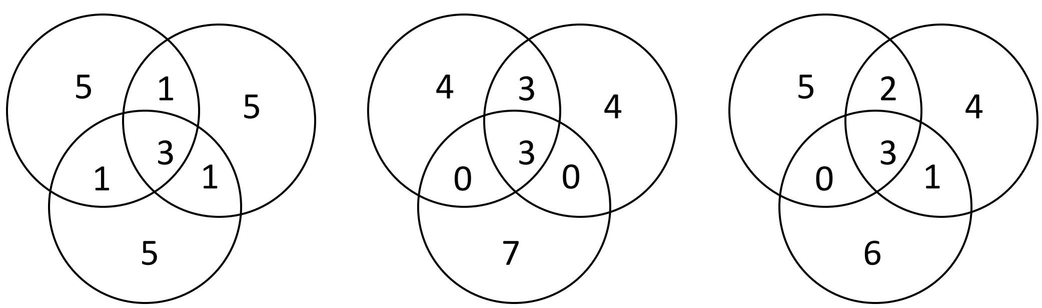
   So the volume of the balloon now is $V = \frac{4}{3} \cdot \pi \cdot R^3 = 5$ and its surface area is $A = 4 \cdot \pi \cdot R^2$. Since the surface area depends on R^2 that means that in order to quadruple it, we need to double the radius.

   Doubling the radius means increasing the volume by a factor of $2^3 = 8$ so the new volume is $5 \cdot 8 = 40$ liters, meaning Danny needs **35** more liters.

2. Though we're not told whether it's the same two formats for the three 2-format books, it does not affect the anwer.

   In each of the three possible scenarios (pictured below), the total number of titles is always $\mathbf{21}$.

   Each circle represents the number of titles in hardcover, paperback, and electronic formats.



Another way of thinking of this is that we can move aside the 3 titles that are in all 3 formats. Now we have 7 titles left in each format. If there were no duplicates, this would be 21 titles. But three titles appear twice, so this set has only 18 distinct titles. Add this to the 3 we put aside at first, and we get 21 overall distinct titles.

3. First we have to pick the 3 books to put on the top shelf, and we have ${}_3C_6 = 20$ ways of doing so. Then on each shelf we have $3! = 6$ permutations of the books, and these are independent of each other, so the total answer is:
${}_3C_6 \cdot (3!)^2 = 20 \cdot 6^2 = \mathbf{720}$ different arrangements.
(This can be seen simply as $6!$ *- we simply order 6 books in a row, and designate the left* 3 *as 'top shelf')*

4. Though this equation is of the 4th degree, only even powers of x appear, so we can make the simple substitution $y = x^2$ to write $y^2 - 13 \cdot y + 36 = 0$
This we can solve to get the solutions $y = 9$ and $y = 4$.
Remembering that $y = x^2$, the x values that solve it are $\pm 2, \pm 3$ and the sum of the *natural* solutions is $2 + 3 = \mathbf{5}$.

5. This problem seems quite complex but can be broken down:
First, how many routes are possible with no limitations? From each city we have 9 options to choose the next destination from, so there are 9^3 three-leg trips possible starting at $E1$. But how many of these visit at least one West coast destination? It's easier to calculate the complement event – how many trips stay on the East coast only? Here we limit our choice of destinations to only 4 possibilities for each leg, so there are 4^3 East-coast-only trips starting from $E1$. Therefore there are $9^3 - 4^3 = \mathbf{665}$ trips that match our criteria.

6. $A - B - \frac{C-E}{D} = 35 - 21 - \frac{720-665}{5} = 35 - 21 - 11 = 3$