Intermediate Mathematics League of Eastern Massachusetts

Category 1 – Mystery

1. At the Science school, there is one microscope for every 5 students.
   The principal figured that in order to have one for every 4 students she needs 6 additional microscopes.
   How many students are at the school?

2. If we continue filling the numbers in the pyramid on the right, which number will be in the rightmost spot on the 12^{th} line?

1

2 3

4 5 6

7 8 9 10

3. On a recent cold morning, two-thirds of the girls and half the boys at school wore gloves. Overall, 60% of students (girls and boys) wore gloves.
   What percent of students are girls?

Answers
1. ________________
2. ________________
3. ____________%

Solutions to Category 1 – Mystery

Answers
1. 120
2. 78
3. 60% (or 60)

1. We can rephrase the question like this:
   For which number the difference between one-fifth and one-quarter is exactly 6?
   Since $\frac{1}{4} - \frac{1}{5} = \frac{5-4}{20} = \frac{1}{20}$, the question becomes:
   One-twentieth of what number is 6? And the answer is obviously $6 * 20 = 120$

2. Looking at the numbers at the rightmost spot of all lines, we can see that the differences between them increase by 1 with each line.
   Therefore, the 12th number in the series will be:
   $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 78$

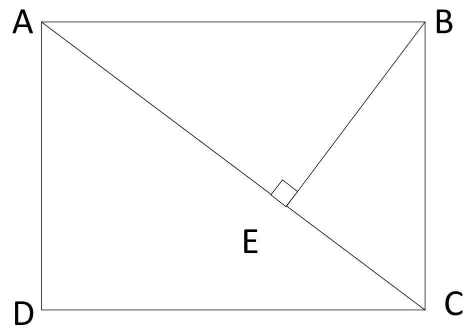
3. If we call the portion of girls (out of all students) G, then the portion of boys is $(1 - G)$.
   Translating the information in the question to an equation we can write:
   $\frac{2}{3} \cdot G + \frac{1}{2} \cdot (1 - G) = 60\% = \frac{3}{5}$.
   To solve we can multiply by 30 and get: $20 \cdot G + 15 \cdot (1 - G) = 18$
   or $5 \cdot G = 3$. Therefore $G = \frac{3}{5} = 60\%$.

Category 2 – Geometry

1. The perimeter of rectangle $ABCD$ measures 70 cm.

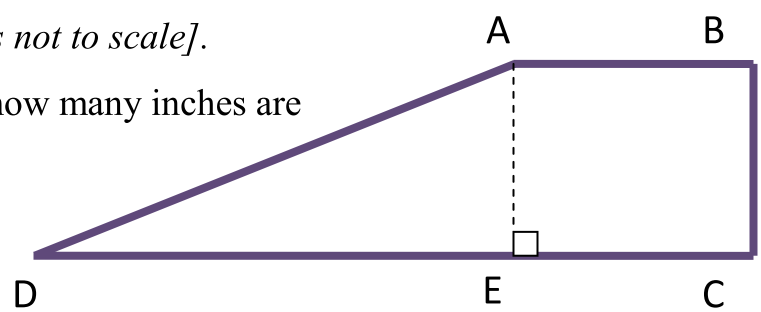
   $\overline{AD} = 15\ cm$, $\overline{AC} = 25\ cm$

   How many centimeters in the measure of $\overline{BE}$?



2. In the drawing below, the area of trapezoid $ABCD$ is four times the area of rectangle $ABCE$. *[The drawing is not to scale]*.

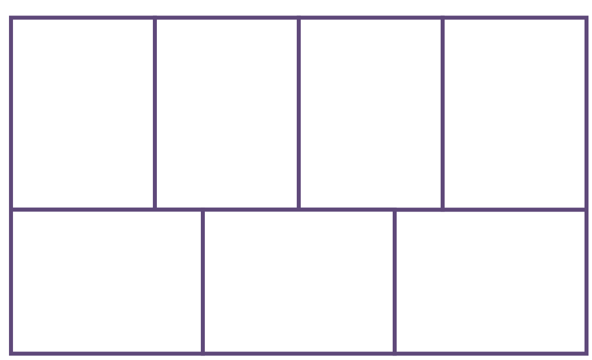
   If $\overline{DC}$ measures 70 inches, then how many inches are there in the measure of $\overline{EC}$?



3. The rectangle below is divided into 7 congruent (identical) rectangles.

   The total area of all is 2,100 square inches.

   How many inches in the perimeter of each one?



Answers

1. ________________

2. ________________

3. ________________

Solutions to Category 2 – Geometery

Answers	
1.	12
2.	10
3.	70

1. For the perimeter to measure 70 cm, $\overline{DC}$ must measure 20 cm. The area of triangle ADC then is $\frac{1}{2} \cdot \overline{AD} * \overline{DC}$ and this must equal the area of triangle ABC which can be expressed as $\frac{1}{2} \cdot \overline{AC} * \overline{BE}$.
   When we plug in the known values we get $\overline{BE} = 12\ cm$.
   [Using similar triangles, you can also observe that $\frac{\overline{AC}}{\overline{AD}} = \frac{\overline{AB}}{\overline{BE}}$].

2. If we call the trapezoid's height $\overline{AE} = \overline{BC} \equiv H$, then we know that the rectangle's area is $\overline{EC} \times H$, and the trapezoid's area is larger by the triangle's area $\frac{1}{2} \times \overline{DC} \times H$. Knowing that $\overline{EC} + \overline{DC} = 70$, and naming $\overline{EC} \equiv L$, we can write: $\frac{1}{2} \times (70 - L) = 3 \times L$ [Expressing the fact that the triangle's area is 3 times the rectangle's, and cancelling out H]. Solving, we get $L = 10\ inches$.

3. Let's call a rectangle's width W, and its height H.
   Each reactangle's area is $W * H = 300\ square\ inches$.
   In the drawing we see 4 rectangles in the top row and 3 in the bottom row, so we can conclude that $4 \cdot W = 3 \cdot H$ or $H = \frac{4 \cdot W}{3}$. When we subtitute this in the first equation we get $\frac{4}{3} \cdot W^2 = 300$ or $W^2 = \frac{900}{4} = 225$.
   So we get $W = 15\ inches\ and\ H = 20\ inches$, and the perimeter is:
   $2 \cdot (W + H) = 70\ inches$.
   [Another way to solve is to notice that the whole area is $3 \cdot H * (H + W) = 2{,}100$. Combined with $W * H = 300$ this leads to $H^2 = 400$].

Category 3 – Number Theory

1. What is the Least Common Multiple of 45 and 66?

2. The Greatest Common Factor (GCF) of two natural numbers A, B is 5, and their product $(A * B)$ is $1{,}000$.
   What is the smallest possible sum of $A + B$?

3. In a far away galaxy, 3 comets are visible from planet 51:
   Comet Alpha is visible every 6 years, and was last seen in 2007.
   Comet Beta is visible every 7 years, and was last seen in 2009.
   Commet Gamma is visible every 8 years, and was last seen in 2009.
   In what year will all three comets be visible together again?

Answers

1. ________________

2. ________________

3. ________________

Answers
1. 990
2. 65
3. 2,121

Solutions to Category 3 – Number Theory

1. $45 = 3^2 * 5$

   $66 = 2 * 3 * 11$

   The Least Common Multiple (LCM) is the product of all prime factors with their highest powers: $LCM(45,66) = 2 * 3^2 * 5 * 11 = 990$

2. The product of two numbers is equal to the product of their GCF and LCM: $A * B = GCF(A,B) * LCM(A,B)$, which in our case is 1,000, so we know that $LCM(A,B) = 1000 \div 5 = 200 = 2^3 \cdot 5^2$. Since $GCF(A,B) = 5$ then only one of the numbers can have 2 as a prime factor, not both of them. Both numbers have 5 as a factor, but only one of them has 5^2 as a factor (otherwise, the GCF would have been 5^2). So either $A = 2^3 \cdot 5^2 = 200 \; and \; B = 5$, or $A = 2^3 \cdot 5 = 40 \; and \; B = 5^2$. The second pair has the smaller sum, 65.

3. For convenience we can deduct 2000 from all years:

   Alpha is visible in years: $7, 13, 19, 25, \ldots$ (remainder 1 when divided by 6)

   Beta is visible in years: $9, 16, 23, 30, \ldots$ (remainder 2 when divided by 7)

   Gamma is visible in years: $9, 17, 25, 33, \ldots$ (remainder 1 when divided by 8)

   So to find the year when all appear, we need to find a number that fits all three conditions. Since (the years for) Alpha and Gamma both should leave a remainder of 1 when divided by 6 or 8, our year should leave a remainder of 1 when divided by any multiple of 6 and 8, and specifically by their LCM, 24. So our year can be one of the numbers $25, 49, 73, 97, 121 \ldots$ but also needs to leave a remainder of 2 when divided by 7. The first number in this series to fit the bill is 121. So our year is 2,121.

Category 4 – Arithmetic

1. Express the decimal 0.425 as a common fraction.

   [A fraction of the form $\frac{m}{n}$ which cannot be simplified].

2. Express the fraction $\frac{7}{48}$ as a decimal.

   Use bar notation to note repeating digits.

3. Tim put all his savings in the Miracle bank.

   After one year, his account's balance grew by 20%.

   After the second year, his balance grew by an additional 25%, and was now $60 more than his original deposit.

   How much money does Tim have now in his account?

Answers

1. ________________

2. ________________

3. ________________

Solutions to Category 4 - Arithmetic

Answers	
1.	$\frac{17}{40}$
2.	$0.1458\overline{3}$
3.	\$180 (or 180)

1. $0.425 = \frac{425}{1000} = \frac{85}{200} = \frac{17}{40}$

2. $0.1458\overline{3}$

   $\overline{7 \mid 48}$
   0
   70
   48
   220
   192
   280
   240
   400
   384
   160
   144
   160 ...

3. If we call his original deposit's amount D, then we can write the information as follows:

   $D + 20\% \cdot D + 25\% \cdot (D + 20\% \cdot D) = D + \$60$

   Replacing percents with numbers and aggregating:

   $D \cdot (1 + 0.2 + 0.25 * 1.2) = D + \$60$ which we can aggregate further into:

   $1.5 * D = D + \$60$ or $D = \$120$.

   The balance now is $1.5 * D = \$180$

Category 5 – Algebra

1. A football thrown at a 45° angle at a speed of $V_{meters/second}$ will travel a horizontal distance of D_{meters}, given by the formula $D = \frac{V^2}{10}$.
   A quarterback throws one ball at a speed of $20_{meters/second}$ and a second ball at a speed of $25_{meters/second}$. How many meters are there between the landing spots of the two balls? *(Both are thrown from the same spot, in the same direction).*

2. The product of three consecutive natural numbers equals fifty-six times their sum. What is the middle number?

3. You lit up a candle at $10:00$ o'clock, and noticed that at $11:00$ o'clock the candle was $\frac{2}{3}$ of the size it was at $10:45$. Assuming the candle burns at a constant rate, at what time will it be gone completely?
   Express you answer in the format HH:MM (Hours:Minutes).

Answers

1. ________________

2. ________________

3. ________________

Solutions to Category 5 - Algebra

Answers
1. $22.5\ or\ 22\frac{1}{2}$
2. 13
3. $11{:}30$

1. Based on our formula, the first ball willt travel a distance of $\frac{20^2}{10} = 40\ meters$, and the second ball will go for $\frac{25^2}{10} = 62.5\ meters$. The difference is $22.5\ meters$.

2. Calling our numbers $x-1, x, x+1$ we can write:
   $(x-1)\cdot x\cdot(x+1) = 56\cdot(x-1+x+x+1)$
   $(x-1)\cdot x\cdot(x+1) = 56\cdot 3\cdot x$
   Dividing both sides by the common factor x we get
   $(x-1)\cdot(x+1) = x^2 - 1 = 168$, the solution to which is 13.
   [Even if you're unsure about the very last step, a little trial and error should help].

3. If it takes M minutes for the candle to burn completely, then by $10{:}45$ the portion of the candle already burnt is $\frac{45}{M}$, and by $11{:}00$ it is $\frac{60}{M}$. So from the question we know that: $\left(1-\frac{60}{M}\right) = \frac{2}{3}\cdot(1-\frac{45}{M})$. To solve, we multiply both sides by M and get $(M-60) = \frac{2}{3}\cdot(M-45)$ and so $M = 90\ minutes$, which brings us to $11{:}30$.
   Another way to think of this is in terms of the rate-of-burn (how much candle is being consumed per minute). If we call this number r then we can write:
   $(1-60\cdot r) = \frac{2}{3}\cdot(1-45\cdot r)$ to get $r = \frac{1\ candle}{90\ minute}$.
   Note that $r = \frac{1}{M}$

Category 6

1. The ratio of a TV set's Height to its Width is called Aspect Ratio.
   The older "Full Screen" TVs had an aspect ratio of 3:4 , and the newer "Wide Screen" TVs have an aspect ratio of 9:16.
   If two TV screens – one old and one new – both are 20 inches wide, then what is the positve difference between their screens' areas? (*Measured in square inches*).

2. What is the fourth (4^{th}) natural number to have exactly 9 factors? (factors include 1 and the number itself).

3. What is the first natural number whose sum-of-factors is exactly 3 times the number itself?

4. As of last September, the U.S. smart-phone operating systems market looked like this:
   1 in every 5 smartphones was Android based, 1 in every 4 was an iphone, and 1 in every 3 was a Blackberry.
   How many phones out of every 100 have an operating system that is different than these three? (*Round your answer to the nearest integer*).

5. Driving on the highway from point A to point B, your car averaged 20 miles-per-gallon. On the way back from B to A, it averaged only 15 miles-per-gallon. Overall your car consumed 28 gallons of gas. How many miles did you drive?

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\frac{A * D}{B + C + E}$$

Answers

1. ____________=*A*
2. ____________=*B*
3. ____________=*C*
4. ____________=*D*
5. ____________=*E*
6. ______________

Solutions to Category 6

Answers	
1.	75
2.	225
3.	120
4.	22
5.	480
6.	2

1. For the "full screen" TV, if the screen is $20''$ wide, then is it $15''$ tall, and so has an area of $15''*20''=300\ square\ inches$. For the "wide screen" TV, a $20''$ wide screen will have a height of $20''*\frac{9}{16}$ and so an area of $20''*20*\frac{9}{16}=25*9=225$ $square\ inches$. The difference is 75 square inches.

2. When we write down the prime factorization of any natural number $N={p_1}^{a_1}\cdot {p_2}^{a_2}\cdot \ldots \cdot {p_s}^{a_s}$ (a product of distinct primes, each raised to the appropriate power), then the number of factors is: $(a_1+1)\cdot(a_2+1)\cdot\ldots\cdot(a_s+1)$. For this product to equal 9, the only configuration is $a_1=a_2=2$, and so the number N is of the form: $N=p^2\cdot q^2=(p\cdot q)^2$ (in other words, N is a perfect square whose root is of the form $p\cdot q$, a product of two distinct primes). The first such number is $N=(2\cdot 3)^2=36$, the second is $N=(2\cdot 5)^2=100$, the third is $N=(2\cdot 7)^2=196$, and the fourth is $N=(3\cdot 5)^2=225$

3. A little trial and error can demonstrate that for most numbers, the sum of factors is less than 3 times the number, and so we're looking for a number whose sum of factors is exceptionally large, so it makes sense to look for a number with a large number of factors. For numbers that are less than 100, the most factors we can have is 12:

$60=2^2\cdot 3\cdot 5\,,\ 72=2^3\cdot 3^2\,,\ 84=2^2\cdot 3\cdot 7\,,\ 90=2\cdot 3^2\cdot 5\,,\ 96=2^5\cdot 3$

But even these nubmers' sum-of-factors is not large enough, so we need to find a number with more factors. Trying to increase the prime factors' powers, the next candidate is $120 = 2^3 \cdot 3 \cdot 5$ whose sum of factors is indeed 360.

[if you're familiar with the sum-of-factors formula you can easily verify:

$\frac{2^4-1}{2-1} * \frac{3^2-1}{3-1} * \frac{5^2-1}{5-1} = \frac{15*8*24}{1*2*4} = 15 * 24 = 360$].

Numbers having this property (sum-of-factors is 3 times the number) are called TriPerfect, and only 6 such numbers are known:

$2^3 \cdot 3 \cdot 5$, $2^5 \cdot 3 \cdot 7$, $2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$, $2^9 \cdot 3 \cdot 11 \cdot 31$,

$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$, $and\ 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$

4. Android has a share of $1\ in\ 5 = 20\%$ of the market.
iphone has a share of $1\ in\ 4 = 25\%$, and Blackberry has a share of $1\ in\ 3 = 33.33\%$. Together the control $20\% + 25\% + 33.33\% = 78.33\%$, and so other competitors have a share of 21.66%.
Out of every 100 phones, 22 will have competing operating systems.

5. If the distance between the two points is D miles, then during the first leg of the trip the car consumed $\frac{D}{20}$ gallons, and during the second leg it consumed $\frac{D}{15}$ gallons, so we know $\frac{D}{20} + \frac{D}{15} = 28 = \frac{7 \cdot D}{60}$. Therefore $D = 240$ miles, and the total distance traveled is $2 \cdot D = 480$ miles.

6. $\frac{A*D}{B+C+E} = \frac{75*22}{225+120+480} = \frac{1650}{825} = 2$