Intermediate Mathematics League of Eastern Massachusetts

Category 1 – Mystery

1. The number of countries in the Americas is the same as the number in Europe. Asia has 3 more countries than Europe, and Africa has 3 more than Asia. Together, these continents have 213 countries.
   How many countries are there in Africa?

2. Starting from the same point at the same time (and going in the same direction), Tom runs at a pace of 5 minutes per mile, and Jerry rides his bike at a pace of 3 minutes per mile.
   What is the distance (how many miles) between them 45 minutes later?

3. October 21st 2010 is a Thursday. What day of the week is October 21st 2012?
   Remember – there are 365 in a year, but 2012 will be a leap year with one extra day in February.

Answers

1. _____

2. _____

3. Circle your answer: Sunday / Monday / Tuesday / Wednesday / Thursday / Friday / Saturday

Answers	
1.	57
2.	6 or 6 miles.
3.	Sunday

Solutions to Category 1 – Mystery

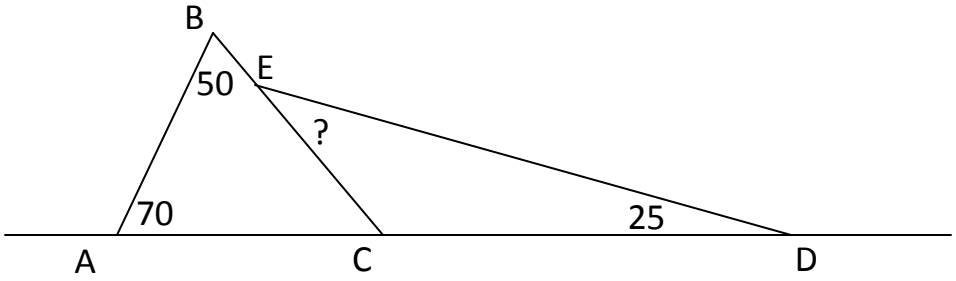
1. If we note the number of countries in the Americas as C, then Europe also has C countries. Asia has $(C+3)$, and Africa has $(C+6)$. Together we know that the total number is:
   $C + C + (C+3) + (C+6) = 4 \cdot C + 9 = 213$
   Therefore $4 \cdot C = 204$ or $C = 51$. Africa has $(C+6) = 57$ countries.
   Algeria , Angola , Benin , Botswana , Burkina Faso , Burundi , Cameroon , Cape Verde , Central African Republic , Chad , Comoros , Democratic Republic of the Congo , Djibouti , Egypt , Equatorial Guinea , Eritrea , Ethiopia , Gabon , Gambia , Ghana , Guinea , Guinea-Bissau , Ivory Coast , Kenya , Lesotho , Liberia , Libya , Madagascar , Malawi , Mali , Mauritania , Mauritius , Mayotte , Morocco , Mozambique , Namibia , Niger , Nigeria , Republic of the Congo , Re'union , Rwanda , Saint Helena , Sao Tome' and Pri'ncipe , Senegal , Seychelles , Sierra Leone , Somalia , South Africa , Sudan , Swaziland , Tanzania , Togo , Tunisia , Uganda , Western Sahara , Zambia , Zimbabwe

2. During the 45 minutes, Tom ran $\frac{45\ minutes}{5\ minutes\ per\ mile} = 9\ miles.$
   Meanwhile, Jerry covered a distance of $\frac{45\ minutes}{3\ minutes\ per\ mile} = 15\ miles.$
   The distance between them is $15 - 9 = 6\ miles.$
   Alternately, we can translate the paces to speed: a pace of 5 minutes per mile is a speed of 12 miles per hour, and a pace of 3 minutes per mile is a speed of 20 miles per hour. At these speeds, Tom ran $12mph \times \frac{3}{4} hour = 9\ miles$, while Jerry biked for $20mph \times \frac{3}{4} hour = 15\ miles$.

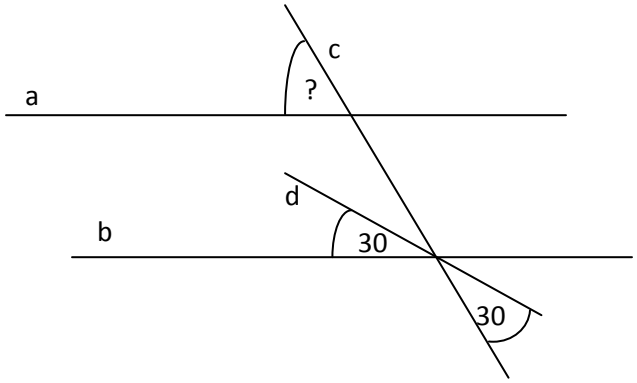
3. Between $10/21/2010$ and $10/21/2012$ we have two full years (one of which is a leap year) for a total of $365 + 366 = 731$ days. So we want to know what is the 731^{st} day after a Thursday. Since days of the week have a weekly (7) cycle, all we care about is the remainder of 731 when divided by 7, which is 3. So the day we're looking for is the 3^{rd} day after Thursday, namely, a Sunday.

Category 2 – Geometry

1. How many degrees are in the measure of angle $\angle CED$?



2. In the drawing below, parallel lines **a** and **b** are both intersected by line **c**. Line **d** passes through the intersection of lines **b** and **c**. Two of the angles are given. How many degrees are in the angle marked by a question mark?



3. How many degrees are in the angle between the hour-hand and the minute-hand of an analog clock when the time is 2:20?

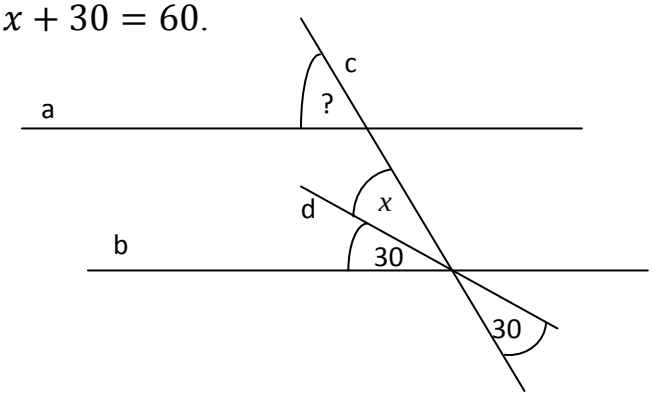
Answers
1. _____
2. _____
3. _____

Solutions to Category 2 – Geometery

Answers	
1.	35
2.	60
3.	50

1. To complete the angles in triangle ABC to 180 degrees, the angle $\angle ACB$ must measure 60 degrees. $\angle ECD$ is its supplement, and so measures 120 degrees. Lastly, $\angle CED$ must complete the angles in triangle ECD to 180 degrees, and so will measure $180 - 120 - 25 = 35$ degrees.

2. The angle in question must equal $(x + 30)$ degrees, as **a** and **b** are parallel (Correspoinding angles). *x* must equal 30 since it is opposite the 'bottom' 30 degree angle in the diagram (so the line **d** bisects the angle between **b** and **c**). So $x + 30 = 60$.



3. Since the dial of a clock is divided to 12 hours, there are $\frac{360}{12} = 30$ degrees between neigboring hours. At 2:20, the minute-hand points to 4, and the hour-hand should be a third of the way between 2 and 3. So the angle between them is equal to $1\frac{2}{3}$ hours or $1\frac{2}{3} * 30\ degrees = 50\ degrees$.

Category 3 – Number Theory

1. Thinking of the natural numbers from 2 to (and including) 20, what is the positive difference between the sum of all composites and the sum of all primes?

2. What value of the digit A will make the number $567{,}88A$ be divisible by 12?

3. Find the sum of all factors of the number 96 (including 1 and 96).

Answers

1. _____

2. _____

3. _____

Solutions to Category 3 – Number Theory

Answers	
1.	55
2.	8
3.	252

1. Sum of primes: $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 = 77$
   Sum of composites:
   $4 + 6 + 8 + 9 + 10 + 12 + 14 + 15 + 16 + 18 + 20 = 132$
   The difference is $132 - 77 = 55$

2. In order to be divisible by 12, a number should be divisible both by 3 and by 4. The sum of digits of $567{,}88A$ is $(7 + A)$, so A can be $2, 5,$ or 8 in order to make it divisible by 3. However, only a value of 8 will make it divisible by 4.

3. The factors of 96 are:
   $96 = 1 \times 96 = 2 \times 48 = 3 \times 32 = 4 \times 24 = 6 \times 16 = 8 \times 12$
   The sum therefore is:
   $1 + 2 + 3 + 4 + 6 + 8 + 12 + 16 + 24 + 32 + 48 + 96 = 252$
   Of all numbers under 100, the number 96 has the largest sum of factors.
   As preparation for meet #2 try to figure out why.

Category 4 – Arithmetic

1. Find the value of the expression:

   $[1 + 2 \div 3 \times 4] \times [9 \times (8 - 7) \div (6 + 5)]$

2. x represents the *mean* value of numbers in the list $\{2, 4, 6, 8\}$.

   y represents the *median* of the numbers in the list $\{0, 2, 4, 6\}$.

   z represents the *mode* of the numbers in the list $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4\}$.

   Find the value of the expression: $(x + y) \div z + (x + z) \div y$

3. Express $\frac{\left(4+4+\sqrt{4}\right)^{\sqrt{4}}}{4\times4-\sqrt{4}}$ as a decimal rounded to the nearest hundredth.

Answers

1. _____

2. _____

3. _____

Solutions to Category 4 - Arithmetic

Answers	
1.	3
2.	5
3.	7.14

1. $[1 + 2 \div 3 \times 4] \times [9 \times (8 - 7) \div (6 + 5)] =$

$$\left[\frac{3}{3} + \frac{2 \times 4}{3}\right] \times \left[\frac{9 \times 1}{(6 + 5)}\right] = \frac{11}{3} \times \frac{9}{11} = \frac{9}{3} = 3$$

2. $x = \frac{2+4+6+8}{4} = 5$.

$y = 3$, the mean of the two middle values $(2\ and\ 4)$.

$z = 4$ as this is the most common value in the set.

Therefore: $(x + y) \div z + (x + z) \div y = (5 + 3) \div 4 + (5 + 4) \div 3 =$

$= 8 \div 4 + 9 \div 3 = 2 + 3 = 5$

3. $\frac{\left(4+4+\sqrt{4}\right)^{\sqrt{4}}}{4\times4-\sqrt{4}} = \frac{(8+2)^2}{16-2} = \frac{100}{14} = 7.\overline{142857} \cong 7.14$

Category 5 – Algebra

1. Find the value of x that solves this equation:
$$\frac{(x-3)}{3}+\frac{x}{4}+\frac{(x-2)}{5}=x-4$$

2. Tom and Jerry are both at home. Tom starts driving in a straight line at 60 miles-per-hour (mph), and 40 minutes later Jerry starts chasing him at 90 mph. How many miles from home will they be when Jerry catches up with Tom?

3. The solution to the equation below is $x = 3$. Find the value of M.
$$(M+1)\cdot x+2\cdot x+5=(x+3)\cdot M-1$$

Answers
1. _____
2. _____
3. _____

Solutions to Category 5 - Algebra

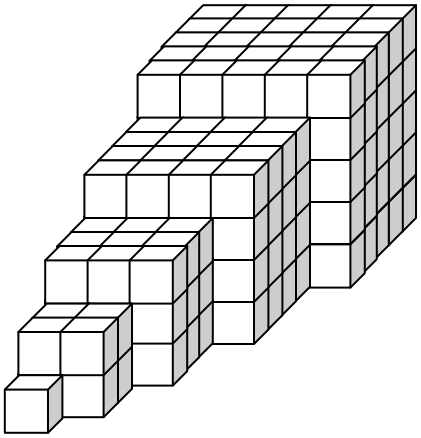
Answers	
1.	12
2.	120
3.	5

1. If we multiply the original equation by the common denominator 60 we'd get:
   $20 \cdot (x-3) + 15 \cdot x + 12 \cdot (x-2) = 60 \cdot (x-4)$
   whice we can aggregate to:
   $20 \cdot x - 60 + 15 \cdot x + 12 \cdot x - 24 = 47 \cdot x - 84 = 60 \cdot x - 240$, or
   $13 \cdot x = 156$ and so $x = \frac{156}{13} = 12$

2. When Jerry catches up with Tom, they both covered the same Distance (D), but Jerry did so 40 minutes ($\frac{2}{3}$ of on hour) more quickly.
   If it took Tom a time T, then we can write the equation:
   $D = 60 * T = 90 * (T - \frac{2}{3})$ $[Distance = Speed * Time]$.
   Solving for T we get $30 * T = 60$ or $T = 2$ $hours$.
   So the distance covered by Tom in 2 hours is 120 miles.

3. Since we know that $x = 3$ is the solution, we can substitute that value for x and then solve for M:
   the equation $(M+1) \cdot x + 2 \cdot x + 5 = (x+3) \cdot M - 1$ will translate to:
   $(M+1) \cdot 3 + 2 \cdot 3 + 5 = (3+3) \cdot M - 1$
   $3 \cdot M + 14 = 6 \cdot M - 1$
   $3 \cdot M = 15$ or $M = 5$

Category 6



1. The drawing on the right shows a tower made of small cubes, lying sideways. If the tower was broken up and all the cubes made a one layer square, how many cubes would fit in the side of that square?

2. What is the median value of all primes that are less than 100?

3. N is a natural number such that the *mean* value of the numbers $\{N, 2, 4, 8\}$ is three times the *median* value. What is the value of N?

4. 32 nations qualified to the 2010 soccer World Cup in South Africa. In the first phase, the teams were divided into 8 groups (of the same size) and in each group each team played all its oponents exactly once. The top 2 teams from each group then advanced to the elimination phase. In the elimination (playoffs) phase, the loser of each game is eliminated from the tournament until only one survives and is crowned the winner. In addition, one more game is played between the losing teams of the semi-finals to determine 3rd and 4th place.
How many games in total were played in the tournament?

5. Starting at 2pm, you watch the hands of an analog clock until the minute-hand points in the opposite direction of the hour-hand (so there are 180 degrees between them).
How many minutes did you have to wait? Round your answer to the nearest integer.

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\frac{(E - B - 1) \times A}{(D - C)}$$

Answers

1. _____=*A*
2. _____=*B*
3. _____=*C*
4. _____=*D*
5. _____=*E*
6. _____

Answers

1. 15
2. 41
3. 58
4. 64
5. 44
6. 5

Solutions to Category 6

1. The drawing shows a lying down tower made of cubes of increasing sizes: 1, 2, 3, 4, and 5 blocks each.
   So overall we have
   $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 1 + 8 + 27 + 64 + 125 = 225$
   cubes. If these were to make a square, it would be 15 blocks wide, as $15^2 = 225$.

2. Listing all primes less than 100, we find there are 25 primes, and so the median value will be the 13th prime (as the first 12 will be of lesser value, and the last 12 will be greater).
   $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \mathbf{41}, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$

3. The mean value is $\frac{N+2+4+8}{4} = \frac{N+14}{4}$ and that is 3 times the median value, so we can write this condition as: $(N + 14) = 12 \times Median\{N, 2, 4, 8\}$.
   The median depends on the value of N:
   If $N \leq 2$ then the median is 3, and then we cannot satisfy the condition because $12 \times Median = 36$ which leads to $N = 22$ in contradiction to $N \leq 2$.
   Similarly, N values less than 8 will not work:

$Value\ of\ N$	1,2	3	4	5	6	7	≥ 8
$Median\{N, 2, 4, 8\}$	3	3.5	4	4.5	5	5.5	6

   Only if we assume $N \geq 8$ then the median is 6 and then:
   $(N + 14) = 12 \times 6 = 72$ and we get $N = 58$

4. In the first phase, we have 8 groups of 4 teams each.
   In each group we need 6 games (if we name the teams A, B, C, D then the games are: $A-B, A-C, A-D, B-C, B-D, C-D$). Multiplied by 8 groups, we get 48 games.
   In the second phase, we start with 16 teams, and each round will eliminate half of the teams (those who lost). The 1st round has 8 games, the 2nd round (quarter finals) has 4 games, the 3rd round (semi-finals) has 2 games, and the finals is the last game. Overall $8+4+2+1=15$ games. With the addition of the game to determine 3rd and 4th place, we have a total of $48+15+1=64$ games.

5. At exactly 2pm, the minute-hand is 60 degrees *behind* the hour-hand, and we want to know when it'll be 180 degrees *ahead* of the hour-hand.
   If the hour-hand travels x degrees by that time, then the minute-head travels $12 \cdot x$ during the same time, and so our requirement is:
   $12 \cdot x - x = 180 + 60$ [The left side of the equation describes how the angle between the hands change, and the right side quantifies what the change needs to be]. The solution is $x = \frac{240}{11}$ degrees, and so the minute-hand has advanced $12 \cdot x = \frac{12 \cdot 240}{11} = 261\frac{9}{11} degrees = 43\frac{7}{11} minutes \cong 44\, minutes.$
   [Recall that for the minute hand, a minute is 6 degrees].

6. $\frac{(E-B-1)\times A}{(D-C)} = \frac{(44-41-1)\times 15}{(64-58)} = \frac{2\times 15}{6} = 5$