Meet #3
January 2009

Intermediate Mathematics League of Eastern Massachusetts

Meet #3
January 2009

Category 1
Mystery
Meet #3, January 2009

1. How many two-digit multiples of four are there such that the number is still a two-digit multiple of four when its digits are reversed?

2. How many different ways can you make $5, if you may use only dimes and quarters and you must use at least one of each?

3. Twelve kids are standing around in a circle to play a game. One of the kids starts by saying the number one, then in a clockwise fashion the kids continue to count by increasing consecutive integers. Each time one of the kids says a prime number he or she sits down and no longer participates in the game. The game continues until only Phil is left standing and he is the winner. What number did Phil say on his first turn?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 1
Mystery
Meet #3, January 2009

Answers

1. 4

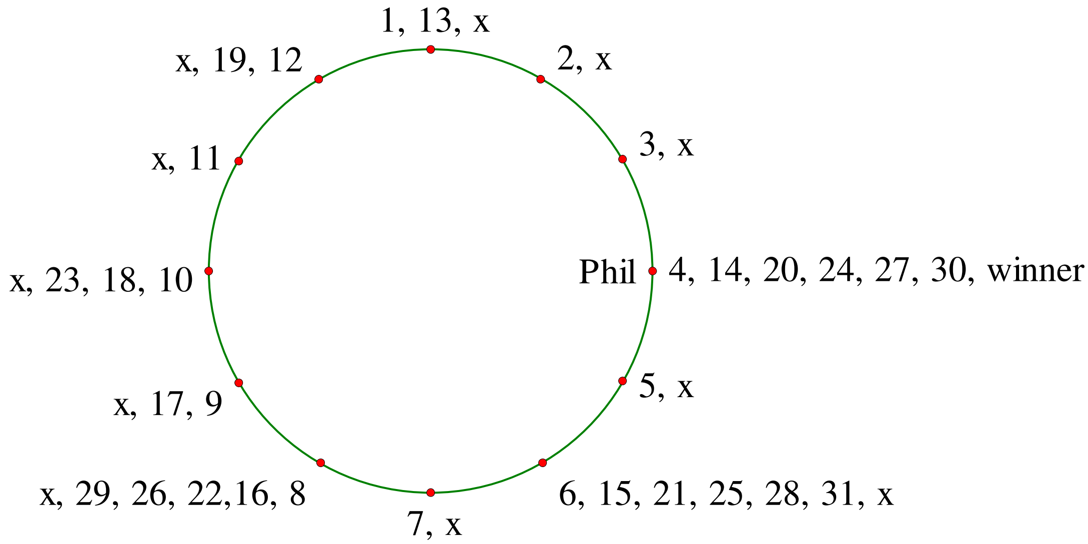
2. 9

3. 4

1. A number is divisible by 4 only if the last 2 digits form a number divisible by 4. All multiples of 4 are clearly even, so switching the digits of a 2 digit multiple of 4 would make the tens digit even. For a 2-digit number with an even tens digit to be divisible by 4 the units digit must be divisible by 4. By that reasoning, the only way that a 2-digit number and the 2-digit number formed by reversing the digits can both divisible by 4 is if both digits are each divisible by 4 themselves. The only numbers that work are 44. 48, 84 and 88, for a total of 4 numbers.

2. If we can only use dimes and quarters, there cannot be an odd number of quarters. Therefore the value of the dimes and the quarters must both be multiples of 50 cents. One way to find the different possible values is to just look at the largest and smallest values. The value of the quarters can be any multiple of 50 cents from 50 cents to $4.50 with the dimes making up the remainder of the $5. That's 9 possible values of the quarters with the rest of the money coming from the dimes.

3. The diagram below shows which numbers each kid says as they count around the circle (the numbers they said are listed in order, moving out from the circle with an "x" shown when they sit down). Phil, the winner, said 4 on his first turn.

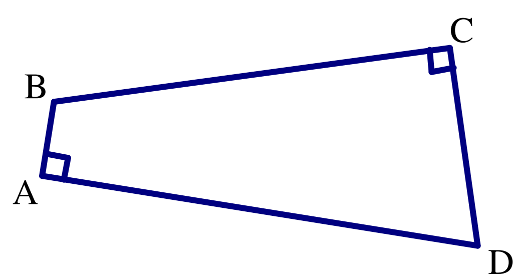


Category 2
Geometry
Meet #3, January 2009

1. How many degrees are in the sum of the interior angles of a convex decagon?

2. Let the number of diagonals in a regular octagon be a, and the number of diagonals in a regular hexagon be b. What is the value of $a - b$?

3. Quadrilateral ABCD has right angles at A and C. The lengths of CD, BC, and AB are 7 cm, 11cm, and 1cm respectively. How many centimeters long is AD?



Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 2
Geometry
Meet #3, January 2009

Answers

1. 1440

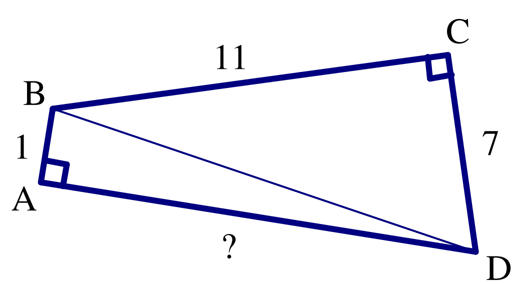
2. 11

3. 13

1. For an n-sided convex polygon, the expression $180(n-2)$ will give the number of degrees in the sum of the interior angles of the polygon. So a decagon $(n = 10)$ would have $180(10-2) = 180(8) = 1440$ degrees.

2. For an n-sided convex polygon, the expression $\frac{n(n-3)}{2}$ will give the number of diagonals in the polygon. So an octagon has $\frac{8(5)}{2} = 20$ diagonals and a hexagon has $\frac{6(3)}{2} = 9$ diagonals. So $a = 20$ and $b = 9$. Therefore $a - b = 20 - 9 = 11$.

3. First draw in segment BD. Since BD is the hypotenuse of triangle BCD, we can use the Pythagorean Theorem to find the length of drawn in segment BD. $\text{BD}^2 = 7^2 + 11^2 = 49 + 121 = 170$. However BD is also the hypotenuse of triangle ABD and therefore $\text{BD}^2 = 1^2 + \text{AD}^2 = 1 + \text{AD}^2 = 170$. So $\text{AD}^2 = 169$ and $\text{AD} = 13$.



Category 3
Number Theory
Meet #3, January 2009

1. Express the base seven number 1234_7 as a base ten number.

$$1234_7 = \underline{\hspace{4cm}}_{10}$$

2. The diameter of the Earth at the equator is approximately 1.28×10^7 meters, while the average diameter of the smallest egg from a creature on Earth, the egg of the *Zenillia pullata*, is approximately 2×10^{-5} meters long. How many times as long as the diameter of the average *Zenillia pullata* egg is the diameter of the Earth at the equator? Express your answer in scientific notation.

3. Find the product of 101101_2 and 22_3 when written in base 2.

$$101101_2 \times 22_3 = \underline{\hspace{3cm}}_2$$

Answers
1. ________________
2. ________________
3. ________________base 2

Solutions to Category 3
Number Theory
Meet #3, January 2009

Answers

1. 466

2. 6.4×10^{11}

3. 101101000_2

1. $1234_7 = 1(7^3) + 2(7^2) + 3(7^1) + 4(7^0) = 1(343) + 2(49) + 3(7) + 4(1) = 343 + 98 + 21 + 4 = 466$

2. To find how many times as long the Earth's diameter is compared to the egg's diameter we need to find the quotient of the two.

$$\frac{\mathbf{1.28 \times 10^{7}}}{\mathbf{2 \times 10^{-5}}} = \frac{\mathbf{1.28 \times 10^{7} \times 10^{5}}}{\mathbf{2}} = \frac{\mathbf{1.28 \times 10^{12}}}{\mathbf{2}} = \mathbf{.64 \times 10^{12}} = \mathbf{6.4 \times 10^{11}}$$

3. Turning both numbers to base 10 first makes the problem $45 \times 8 = 360$. Turning 360 back to base 2 yields 101101000_2.

A more elegant way to do this is to notice that $22_3 = 8 = 1000_2$. Multiplying a base 2 number by 1000_2 has the same effect as multiplying a base 10 number by 1000. You just need to add 3 zeros to the end of the original number!

Category 4
Arithmetic
Meet #3, January 2009

1. What is the largest possible integer value of a if $3^a < 2^{13}$?

2. What is the value of the expression below? Express your answer as a common fraction.

$$\left(3\frac{1}{8}\right)^5 \times \left(1\frac{1}{4}\right)^{-7}$$

3. There are exactly 38 whole numbers that are greater than $\sqrt[3]{400}$ and less than $\sqrt{n}$. If n is a whole number, what is the largest possible value of n?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 4
Arithmetic
Meet #3, January 2009

Answers

1. 8

2. $\frac{125}{2}$

3. 2116

1. $2^{13} = 8192$. The largest power of 3 less than that is $3^8 = 6561$, so a = 8.

2. $\left(3\frac{1}{8}\right)^5 \times \left(1\frac{1}{4}\right)^{-7} = \left(\frac{25}{8}\right)^5 \times \left(\frac{5}{4}\right)^{-7} = \left(\frac{25}{8}\right)^5 \times \left(\frac{4}{5}\right)^7 =$

$$\frac{25^5 \times 4^7}{8^5 \times 5^7} = \frac{5^{10} \times 2^{14}}{2^{15} \times 5^7} = \frac{5^3}{2} = \frac{125}{2}$$

3. $7^3 = 343$ and $8^3 = 512$, so $\sqrt[3]{400}$ is between 7 and 8. The first whole number greater than $\sqrt[3]{400}$ then is 8. The 38th whole number greater than $\sqrt[3]{400}$ is 45. Therefore $\sqrt{n}$ must be larger than 45, but not larger than 46 or there would be 39 numbers between $\sqrt[3]{400}$ and $\sqrt{n}$. $45^2 = 2025$ while $46^2 = 2116$. So the largest whole number value of n would be 2116 as 45 is less than $\sqrt{2116}$ while 46 is not.

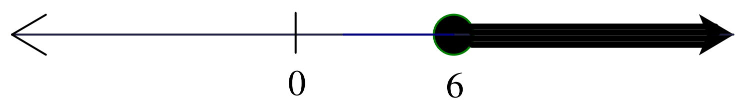
Category 5
Algebra
Meet #3, January 2009

1. What is the least possible solution for x in the inequality below?

$$|8 - 2x| \leq 16$$

2. For what value of K does the solution to the inequality below match the graph below?

$$x(K - 5) + 12 \leq K - 8$$



3. Mike chose 3 distinct numbers from the set below and took the absolute value of their sum. Sean chose 3 distinct numbers (not necessarily different from Mike's) from the set below and took the sum of their absolute values. What is the greatest possible absolute value of the difference between Mike's and Sean's result?

$$\{-12, -7, -2, 1, 3, 6, 11\}$$

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #3, January 2009

Answers

1. -4

2. 2

3. 30

1. $|8-2x| \leq 16$

$-16 \leq 8-2x \leq 16$

$-24 \leq -2x \leq 8$

$12 \geq x \geq -4$ So – 4 is the least integer solution for x.

2. $x(K-5)+12 \leq K-8$

$x(K-5) \leq K-20$

$x \leq \frac{K-20}{K-5}$

Because we've divided by $(K-5)$ if $(K-5)$ is a negative number, we would have to reverse the $\leq$ to $\geq$. According to the graph, the solution is $x \geq 6$ which means the inequality was reversed and $(K-5)$ must be negative. Therefore we actually have $x \geq \frac{K-20}{K-5}$ and since we know $x \geq 6$ the equation below finds K.

$6 = \frac{K-20}{K-5}$

$6(K-5) = K-20$

$6K-30 = K-20$

$5K = 10$

$K = 2$

3. To get the largest absolute value of a sum Mike wants to pick numbers that are all positive or all negative so that he is adding their values. The largest he could get would be $|(-12)+(-7)+(-2)| = |-21| = 21$. The smallest value Mike could get would be $|(-12)+1+11| = 0$ or $|(-7)+1+6| = 0$.

For Sean to get the largest sum of the absolute values he just needs the three numbers with the largest absolute values. In this case he wants $-12, 11$ and -7 for which the sum of their absolute values is $|-12|+|11|+|-7| = 12+11+7 = 30$. The smallest value he could get would be $|-2|+|1|+|3| = 6$.

The greatest possible difference between Mike's result and Sean's result would be by taking Sean's largest (30) and Mike's smallest (0). $30-0=30$.

Category 6
Team Questions
Meet #3, January 2009

1. If $\left(\sqrt[5]{x}\right)^3 = 8$, $\left(\sqrt[3]{y}\right)^{-2} = 9$ and $\left(\sqrt[4]{z}\right)^5 = 243$, what is the value of xyz?

2. The average of the first n positive square numbers is 46. What is the value of n?

3. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, 13, 21… where each new term (after the first 2) is found by taking the sum of the previous two terms. The first term, 1, shall be called F_1, and the second term, also 1, shall be called F_2. For what positive value of n, with $n > 1$, will both F_n and F_{n+1} have units digits both equal to 1?

4. Three regular polygons, not necessarily distinct, have a total of 76 diagonals. What is the greatest total number of sides the three polygons could have?

5. Of all the positive integers that can be written in base 3 without using zero as a digit, what is the 20th smallest positive integer? Give your answer in base ten.

Answers

1. _______________ = A
2. _____________ = B
3. ______________ = C
4. ______________ = D
5. _____________ = E
6. __________________

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below.

$$\sqrt{\frac{E}{\sqrt{\frac{A}{C-(B+D)}}}}$$

Solutions to Category 6
Team Questions
Meet #3, January 2009

Answers

1. 96

2. 11

3. 61

4. 26

5. 50

6. 5

1. $\left(\sqrt[5]{x}\right)^3 = 8 \rightarrow \left(\sqrt[5]{x}\right)^3 = 2^3 \rightarrow \sqrt[5]{x} = 2 \rightarrow x = 32$

$\left(\sqrt[3]{y}\right)^{-2} = 9 \rightarrow \left(\sqrt[3]{y}\right)^{-2} = \left(\frac{1}{3}\right)^{-2} \rightarrow \sqrt[3]{y} = \frac{1}{3} \rightarrow y = \frac{1}{27}$

$\left(\sqrt[4]{z}\right)^5 = 243 \rightarrow \left(\sqrt[4]{z}\right)^5 = 3^5 \rightarrow \sqrt[4]{z} = 3 \rightarrow z = 81$

So $xyz = 32\left(\frac{1}{27}\right)(81) = 96$

2. Making a chart with the sum of the squares and the quotient of the sums divided by n yields the following table. Since the question is asking for the average of the first n square numbers, if the average is 46 we don't need to even begin the table until $n = 7$, the first square larger than 46.

n	Sum of 1st n squares	Sum divided by n
7	140	20
8	204	25.5
9	285	31.666…
10	385	38.5
** 11 **	506	** 46 **

Using the formula for the sum of the square numbers this problem can be done without the table. For the sum, S, we have the sum of the first n squares as:

$S = \frac{n(n+1)(2n+1)}{6}$ Since we know the average, if we divide the right by n we get:

$46 = \frac{n(n+1)(2n+1)}{6n} \rightarrow 276 = (n+1)(2n+1) \rightarrow 276 = 2n^2 + 3n + 1 \rightarrow$

$0 = 2n^2 + 3n - 275 \rightarrow 0 = (n-11)(2n+25)$

$0 = n - 11$ or $0 = 2n + 25$

$n = 11$ or $n = -12.5$

So $n = 11$

3. Listing the units digits of the terms of the Fibonacci sequence in groups of ten yields:

1, 1, 2, 3, 5, 8, 3, 1, 4, 5,
9, 4, 3, 7, 0, 7, 7, 4, 1, 5,
6, 1, 7, 8, 5, 3, 8, 1, 9, 0,
9, 9, 8, 7, 5, 2, 7, 9, 6, 5,
1, 6, 7, 3, 0, 3, 3, 6, 9, 5,
4, 9, 3, 2, 5, 7, 2, 9, 1, 0, 1, 1, 2, ……

The next time the units digits of F_n and F_{n+1} are both equal to 1 is when $n = 61$.

4. Start by determining how many diagonals are in each of the regular polygons for polygons which have less than 76 diagonals.

Listed consecutively for a square through a regular 13-gon, the number of diagonals in each is: 2, 5, 9, 14, 20, 27, 35, 44, 54, 65

There are 4 ways in which 3 of those diagonal totals can have a sum of 76:

# of diagonals		# of sides
65 + 9 + 2 = 76	→	13 + 6 + 4 = 23
54 + 20 + 2 = 76	→	12 + 8 + 4 = 24
44 + 27 + 4 = 76	→	11 + 9 + 5 = 25
35 + 27 + 14 = 76	→	10 + 9 + 7 = 26

The maximum number of sides in total for the 3 polygons is 26.

5. Counting in base 3 without using any zeros results in the following list:

1, 2, (note: 2 numbers)
11, 12, 21, 22, (note: 4 numbers)
111, 112, 121, 122, 211, 212, 221, 222, (note: 8 numbers)
1111, 1112, 1121, 1122, 1211, 1212 (20^{th} number in list)

$1212_3 = 1(27) + 2(9) + 1(3) + 2(1) = 27 + 18 + 3 + 2 = 50$

6. $$\sqrt{\frac{E}{\sqrt{\frac{A}{C-(B+D)}}}} = \sqrt{\frac{50}{\sqrt{\frac{96}{61-(11+26)}}}} = \sqrt{\frac{50}{\sqrt{\frac{96}{24}}}} = \sqrt{\frac{50}{\sqrt{4}}} = \sqrt{\frac{50}{2}} = \sqrt{25} = 5$$