Meet #5
March 2008

Intermediate Mathematics League of Eastern Massachusetts

Meet #5
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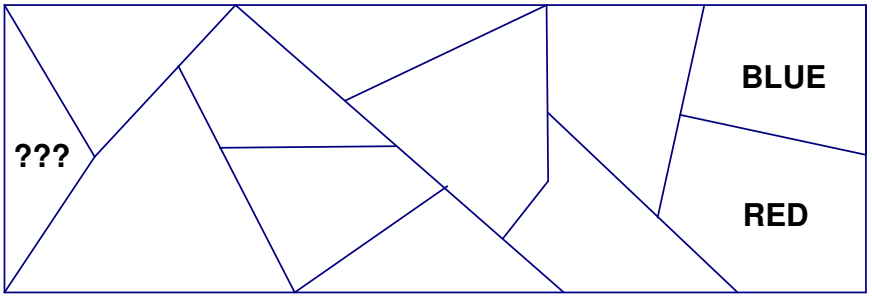
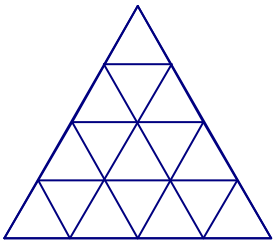
Category 1
Mystery
Meet #5, March 2008

1. In the diagram to the right, each non-overlapping section of the large rectangle is painted either RED, BLUE, or GREEN. No two sections that share an edge can be painted the same color. Two of the sections are labeled with the colors they have been painted already. What color would the section with the question marks (???) be painted? (Two sections that share a vertex could have the same color if they only shared that one point)



2. How many degrees are in the angle formed by the hands of a normal 12-hour clock at 9:30am if the angle we want is less than 180 degrees?



3. How many triangles of any size can be found in the picture to the left?

Answers

1. _____
2. _____
3. _____

Solutions to Category 1
Mystery
Meet #5, March 2008

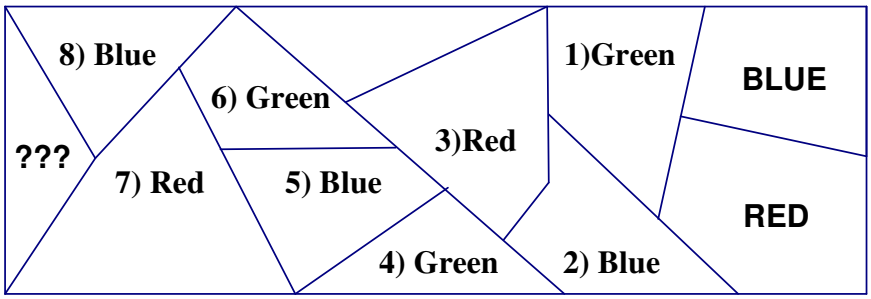
Answers

1. Green

2. 105

3. 27

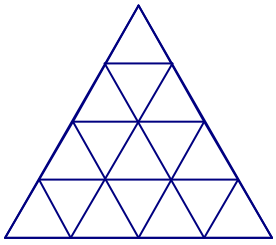
1. The diagram below explains how the picture would have to be colored. The order of the steps taken are given by number in front of the color name. Notice there is one unpainted section. That section would be blue, but is not necessary for the solution.



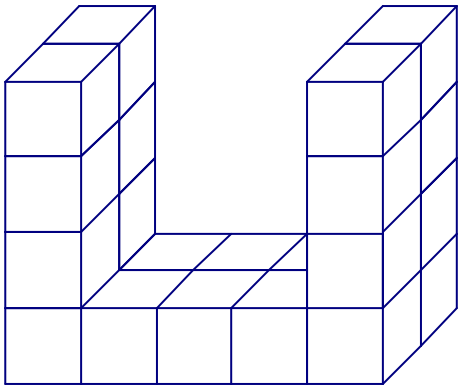
So the section with the question marks (???) must be **Green**

2. At 9:30 the minute hand is pointed at the 6 and the hour hand is exactly half way between the 9 and 10. There are 30 degrees between any 2 consecutive numbers of the clock face, so there are :
$30 \times 3\frac{1}{2} = \mathbf{105}$ degrees in the angle between the hands.

3. There are 16 triangles that consist of one of the smallest triangles. There are 7 triangles that consist of 4 of the smallest triangles (one is upside down). There are 3 triangles that consist of 9 of the smallest triangles. There is 1 triangle that is the entire diagram. That is a total of 16 + 7 + 3 + 1 = **27** triangles.

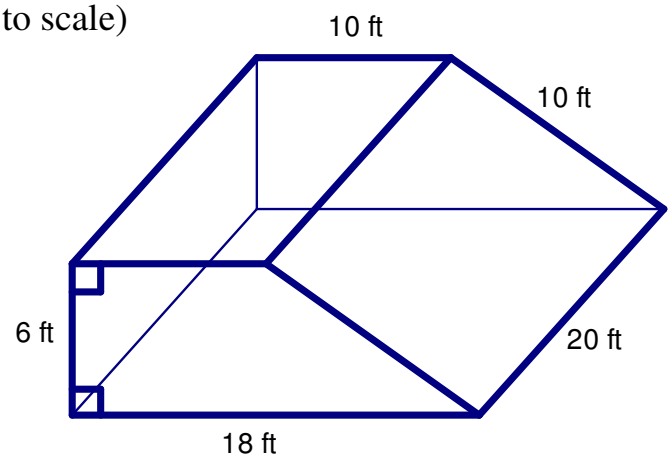


Category 2
Geometry
Meet #5, March 2008

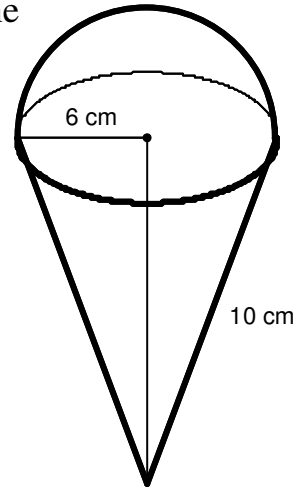


1. Twenty-two unit cubes were used in building the "U" shape at the right. How many square units are in the surface area of this figure? (Note : The back view is the same as the front.)

2. How many square feet are in the surface area of the right trapezoidal prism below? (diagram not drawn to scale)



3. The diagram to the right shows a hemisphere sitting on top of a cone. The circular base of the hemisphere and the circular base of the cone have the same area. The radius of the hemisphere is 6cm and the slant height of the cone is 10 cm. How many cubic centimeters are in the volume of the combined shape? Use 3.14 as an estimation of π and express your answer to the nearest tenth.



Answers

1. _____
2. _____
3. _____

Solutions to Category 2
Geometry
Meet #5, March 2008

Answers

1. 70

2. 1048

3. 753.6

1. Viewing from the front there are 11 visible squares, and same from the back. From both the left and right sides there are 8 viewable squares. From both the top and bottom there are 10 viewable squares. There are an additional 12 viewable squares inside on the left and right faces of the "U" shape That's a total of 11+11+8+8+10+10+12 = **70** square units.

2. There are 6 faces to this prism. The top has area 10x20=200. The bottom has area 18x20=360. The left face has area 6x20=120. The right face has area 10x20=200. The two trapezoids have area $\frac{(10+18)\times 6}{2} = 84$. The total surface area is 200+360+120+200+84+84 = **1048**

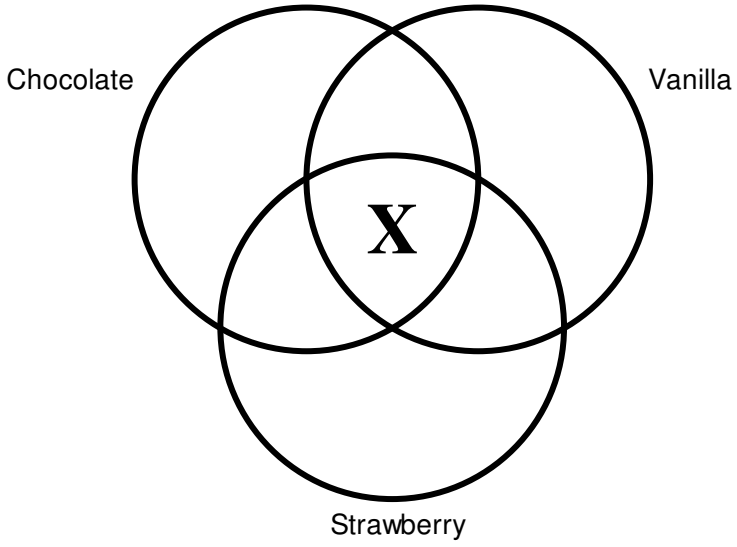
3. The volume of the hemisphere is $\frac{1}{2} \times \frac{4}{3}\pi 6^3 = \frac{2}{3}\pi 216 = 144\pi \approx 452.16$ Use the Pythagorean Theorem to find that the height of the cone is 8 cm. So the volume of the cone is $\frac{1}{3}\pi 6^2 \times 8 = \frac{1}{3}\pi 36 \times 8 = \pi 96 \approx 301.44$. The combined volume is 452.16 + 301.44 = **753.6**

Category 3
Number Theory
Meet #5, March 2008

1. There are 30 students in Mike's math class. Seventeen of those students take Spanish, 18 of them take Chorus, and 6 of them take neither. How many of the students in Mike's math class take both Spanish and Chorus?

2. For this problem consider the numbers from 1 to 50 inclusive only. Set A consists of the positive integers that are one greater than a multiple of 3, set B consists of the positive integers that are one greater than a multiple of 4, and set C consists of the perfect square numbers. How many elements are in $C \cup (A \cap B)$?
(reminder : $\cup$ means union and $\cap$ means intersection)

3. Eighty people were polled to find out what flavor of ice cream they like most from the choices of Chocolate, Vanilla, or Strawberry. Each person was given the diagram below and asked to put an "X" in the spot that best described what flavor or flavors they like the most. Sixty people placed their "X" somewhere in the chocolate circle, forty put the "X" in the Vanilla circle, and thirty put the "X" in the strawberry circle. If twenty people put the "X" in a spot that said they liked two of the flavors but not the third, how many people placed their "X" the same way it was done in the diagram below? (Everyone liked at least one of the flavors.)



Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #5, March 2008

Answers

1. 11

2. 9

3. 15

1. If 17 take Spanish and 18 take Chorus, it would appear that there are 35 kids. However, there are only 30 kids and 6 who take neither, so only 24 kids in these classes. Where did those extra 11 kids come from? They came from **11** kids taking both classes and were counted twice in the original count of 35.

2.

A = {1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49}
B = {1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49}
C = {1, 4, 9, 16, 25, 36, 49}

$A \cap B$ = { 1, 13, 25, 37, 49} (5 elements)
*each is 1 more than a multiple of 12

Set C has 7 elements, but has 3 elements in common with $A \cap B$ {1, 25, 49}, so the union of the two sets has 5 + 7 – 3 = **9** elements as listed below:

$$C \cup (A \cap B) = \{1, 4, 9, 13, 16, 25, 36, 37, 49\}$$

3. If 60 people like chocolate, 40 like Vanilla, and 30 like Strawberry that would appear to be 130 people. We know there were only 80 people polled so some of them were counted for more than 1 flavor. We also know that 20 people like 2 flavors, so that means 20 were counted twice, so we are down to 130 – 20 = 110 people, but still 30 more than there actually were. Any person that put their X in the center was counted 3 times which is 2 extra each. If we have 30 extra from that center region it must be because there were 30÷2 = **15** people who put their X in the center and like all 3 flavors.

Category 4
Arithmetic
Meet #5, March 2008

1. Two standard six sided dice are rolled. What is the probability that the sum of the numbers on the top faces of the dice is a prime number? Give your answer as a simplified fraction.

2. Arjun averaged exactly 93% on nine quizzes. Arjun's teacher decided to drop each student's highest and lowest quiz grades. After dropping the scores Arjun's average increased to a 97%. What is the average of the two quiz scores that were dropped?

3. At the Institute of Math Learning and Extra Mathematics there are 24 students in the honors math class. Fourteen of the students are boys. Two boys and two girls are randomly chosen from the class to speak at the 8^{th} grade graduation. How many different groups of four students can be chosen if there must be two boys and two girls?

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #5, March 2008

Answers

1. 5/12

2. 79

3. 4095

1. Using the table below you can see that 15 out of 36 of the sums are prime. So the probability is $\frac{15}{36} = \mathbf{\frac{5}{12}}$

+	1	2	3	4	5	6
1	**2**	**3**	4	**5**	6	**7**
2	**3**	4	**5**	6	**7**	8
3	4	**5**	6	**7**	8	9
4	**5**	6	**7**	8	9	10
5	6	**7**	8	9	10	**11**
6	**7**	8	9	10	**11**	12

2. If he averaged 93% on the first 9 quizzes, then the total of the 9 quizzes is 93 x 9 = 837. After dropping 2 of the scores the teacher is only counting 7 quizzes and the total would be 97 x 7 = 679 points. That means the total of the 2 quizzes dropped is 837 – 679 = 158 points. If the sum of those 2 is 158, the average is $\frac{158}{2} = \mathbf{79}$.

3. First we need to choose the 2 boys which can be done in ${}_{14}C_2 = \frac{14\times13}{2} = 91$ ways. Next choose the 2 girls which can be done in ${}_{10}C_2 = \frac{10\times9}{2} = 45$ ways. Those two decisions can happen in a total of $91 \times 45 = \mathbf{4095}$ ways.

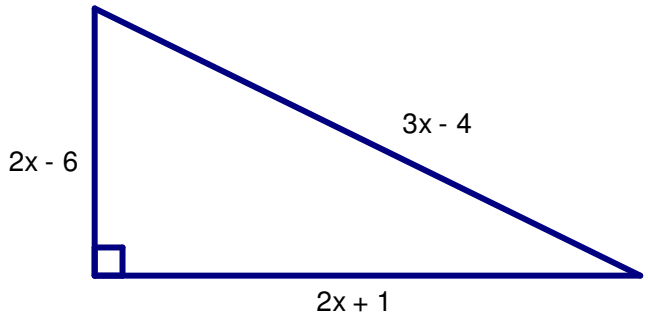
Category 5
Algebra
Meet #5, March 2008

1. What is the sum of the *x*-coordinates of the *x*-intercepts of the equation below? (the x-intercepts are known as the roots and are the values of x when y = 0.)

$$y = 2x^2 - 2x - 12$$

2. A square is changed into a new rectangle by adding 3 cm to the length and subtracting 5 cm from the width. The area of the new rectangle is 48 cm. How many square centimeters are in the area of the original square?

3. The triangle below is a right triangle with hypotenuse of length (3x – 4). What is the value of the perimeter of this triangle?



Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #5, March 2008

Answers

1. 1

2. 81

3. 40

1.

$y = 2x^2 - 2x - 12$
$y = (2x + 4)(x - 3)$
$0 = (2x + 4)(x - 3)$
$2x + 4 = 0$ or $x - 3 = 0$
$x = -2$ or $x = 3$
The sum is $-2 + 3 = \mathbf{1}$

Another way to find the sum of the x-coordinates of the x-intercepts (otherwise known as roots) is to put the equation into the form $y = ax^2 + bx + c$ and then the sum is $\frac{-b}{a} = \frac{-(-2)}{2} = \mathbf{1}$

2. If the square's original side length is *s* then the new rectangle has length *s* +3 and width *s* – 5 and the area would be :
$(s + 3)(s - 5) = 48$
$s^2 - 2s - 15 = 48$
$s^2 - 2s - 63 = 0$
$(s - 9)(s + 7) = 0$
$s - 9 = 0$ or $s + 7 = 0$
$s = 9$ or $s = -7$ but since *s* is a side of a square, it can't be negative. So *s* = 9 and the area of the square was 9^2 **=81**

3. Using the Pythagorean Theorem, we know that :
$(2x - 6)^2 + (2x + 1)^2 = (3x - 4)^2$
$4x^2 - 24x + 36 + 4x^2 + 4x + 1 = 9x^2 - 24x + 16$
$8x^2 - 20x + 37 = 9x^2 - 24x + 16$
$0 = x^2 - 4x - 21$
$0 = (x - 7)(x + 3)$
$0 = x - 7$ or $0 = x + 3$
$x = 7$ or $x = -3$ but if $x = -3$ then all the sides would be negative which we can't have, so $x = 7$. That makes the three sides of the triangle 8, 15, and 17 and the perimeter is **40**.

Category 6
Team Questions
Meet #5, March 2008

1. A rectangular prism is made up of many unit cubes. The dimensions of the prism are 32 blocks long by 9 blocks wide by 6 blocks tall. The prism is taken apart and all the unit cubes are reassembled into a new prism with the smallest possible surface area. What is the sum of the length, width, and height of this new prism?

2. What is the probability that a family of 5 children has at least 1 boy? (Assume the probability of a child being a boy is equal to the probability of it being a girl and that there were no twins, triplets, or "multiple births" at one time in this family.) Give your answer as a simplified fraction.

3. In the line segment $\overline{AF}$ below, line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EF}$ are all congruent and have a length equal to 1 unit. What is the length of the segment defined by :

$$[(\overline{AD} \cap \overline{BF}) \cup (\overline{BE} \cap \overline{DF})] \cap \overline{AD} \ ?$$

4. In order for a lever to be balanced, the product of the distance from the balance point (triangle in picture) and the weight of the item at the end of the lever must be equal on both sides. (The weight is indicated in the box at the end of the lever and the distance is given under the lever.) Find the value of X so that this lever is balanced and the weight on each end is greater than zero.

5. In the magic square to the right, the center square is shaded in and not used as part of the magic square. The numbers 1 through 8 are to be placed in the 8 small unshaded squares such that the sum of the numbers in each of the 4 rows and columns will all be the same and that sum will be as large as possible. If the number 3 is placed as shown, what is the largest possible sum of the numbers that could be placed in the boxes labeled A and B?

3	B	
A		

Answers

1. _____ = *A*
2. _____ = *B*
3. _____ = *C*
4. _____ = *D*
5. _____ = *E*
6. _____

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below.

$$128B + \frac{\sqrt{AE+37} - 1}{CD}$$

Solutions to Category 6
Team Questions
Meet #5, March 2008

Answers

1. 36

2. 31/32

3. 2

4. 3

5. 9

6. 127

1. A prism with dimensions 32 by 9 by 6 would use 1728 unit cubes. The prism with the least surface area would be the one that was closest to a cube, in other words with dimensions closest to each other. Since 1728 = 12^3 we can make the prism a cube with side length 12. The sum of the dimensions is 12 + 12 + 12 = **36**.

2. Five children could have been born in $2^5 = 32$ different ways. Only 1 of those ways would not have any boys, the 1 that is all girls. So the probability that there is at least one boy is $\mathbf{\frac{31}{32}}$**.**

3.

$$[(\overline{AD} \cap \overline{BF}) \cup (\overline{BE} \cap \overline{DF})] \cap \overline{AD}$$
$$[(\overline{BD}) \cup (\overline{DE})] \cap \overline{AD}$$
$$\overline{BE} \cap \overline{AD}$$
$$\overline{BD}$$

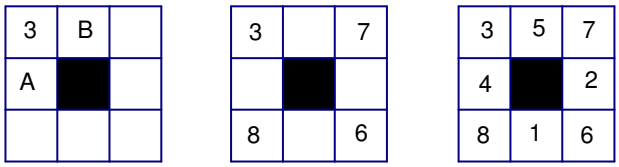
Since the points are all 1 unit apart from the next point, $\overline{BD}$ is 2 units long.

4. $(2x + 2)(x + 3) = (3x + 3)(x + 1)$
$2x^2 + 8x + 6 = 3x^2 + 6x + 3$
$0 = x^2 - 2x - 3$
$0 = (x - 3)(x + 1)$
$0 = x - 3$ or $0 = x + 1$
$x = 3$ or $x = -1$ but x can't equal -1 since the weight on the right side would be zero, so x= **3**

As an alternate solution, if you notice that the weight on the left is twice the weight on the right, then the length on the right must be twice the length on the left :

$3x + 3 = 2(x + 3) \rightarrow 3x + 3 = 2x + 6 \rightarrow x = 3$

5. If we want the sum to be as large as possible we need to put all the largest numbers in the corners where they would be counted twice. So we would use 6, 7, and 8 in the three remaining corners. Now that we know which numbers are being counted twice, we can figure out what the sum will be in each row. Each number 1 – 8 will be used and the sum of the digits from 1 to 8 is 36. Remember also that we will be counting 3, 6, 7, and 8 twice since they are in the corners and therefore in two rows/columns. Adding those to our total we get 36+3+6+7+8 = 60. That total of 60 comes from 4 rows and columns, so each would have a sum of 15. If the sum is 15 we know 8 and 7 can not be in the same row since we have no 0 to use. Placing 6 across from the 3 (diagonally) and the 7 and 8 in opposite corners forces the other 4 squares to be filled in an exact way so that each sum is 15. The three steps of filling in the squares is shown here :



So A + B = 4 + 5 = **9** Note that the numbers could all be reversed if 8 & 7 are switched, but the sum of A and B would not change.

6. $128\left(\frac{31}{32}\right) + \frac{\sqrt{36\times9+37}-1}{2\times3}$

$$4(31) + \frac{\sqrt{324+37}-1}{6}$$

$$124 + \frac{\sqrt{361}-1}{6} = 124 + \frac{19-1}{6} = 124 + \frac{18}{6} = 124 + 3 = \mathbf{127}$$