Meet #2
November 2007

Intermediate Mathematics League of Eastern Massachusetts

Meet #2
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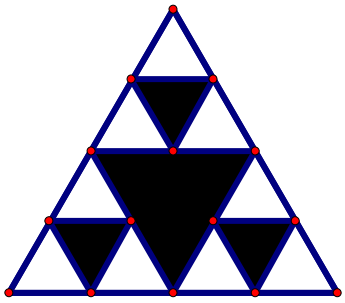
Category 1
Mystery
Meet #2, November 2007

1. Han and Sean are playing a game. Han tells Sean to think of a number. Han then tells Sean to perform these computations, in the given order, to get a new number.

Multiply by six.
Add eighteen.
Divide by two.
Add twelve.
Divide by 3.
Subtract his original number.
After these computations, what is Sean's new number?

2. In the diagram to the right, the midpoints of the sides of the largest triangle are connected to form a new triangle, and that triangle is shaded. The process is then repeated in the three triangles that were not shaded. What fraction of the largest triangle is now shaded?



3. Billy has a bag of several coins, none larger in value than a quarter. He notices that he has the exact same number of each type of coin that he has, although he has either two or three different types of coins. If he has a total of $6.15, how many coins does he have all together?

Answers
1. _____
2. _____
3. _____

Solutions to Category 1
Mystery
Meet #2, November 2007

Answers

1. 7

2. $\frac{7}{16}$

3. 82

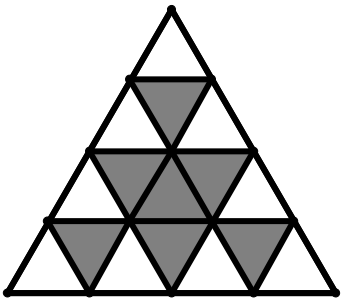
1. $(6x+18)\div 2 \Rightarrow 3x+9$

$3x+9+12 \Rightarrow 3x+21$

$(3x+21)\div 3 \Rightarrow x+7$

$(x+7)-x \Rightarrow 7$

2. By adding the extra lines to the center triangle, it is clear that 7 out of the 16 small triangles are shaded, so $\frac{7}{16}$ of the large triangle is shaded.



You can also look at this by saying that in the first step $\frac{3}{4}$ is left white. In the second step $\frac{3}{4}$ of that $\frac{3}{4}$ is left white. So $\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$ is left white and $1-\frac{9}{16}=\frac{7}{16}$ has been shaded.

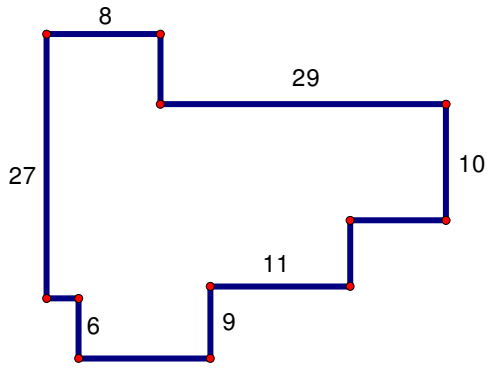
3. Since Billy has either 2 or 3 different types of coins and he has the same number of each type that he does have, his total amount of money must be a multiple of one of these totals :

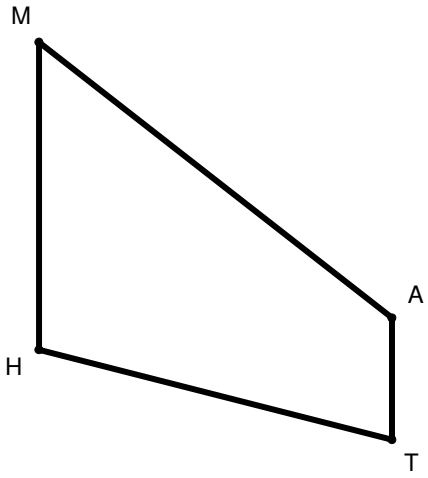
1+5	1+10	5+10	1+5+10	1+25	5+25	1+5+25	10+25	1+10+25	5+10+25
6	11	15	16	26	30	31	35	36	40

Since 615 is odd, that eliminates 6, 16, 26, 30, 36 and 40. Of 11, 15, 31 and 35 the only one that is a factor of 615 is 15(nickel plus a dime). 615 ÷ 15 = 41, so there are 41 nickels and 41 dimes for a total of **82** coins.

Category 2
Geometry
Meet #2, November 2007

1. In the figure to the right, all angles are right angles. What is the perimeter of the figure?





2. Quadrilateral MATH to the left has sides MH and AT parallel to each other with MH being three times as long as AT. If the shortest distance between the parallel sides is 5 inches and the area of the quadrilateral is 15 in^2, how many inches long is side AT? Express your answer as a decimal.

3. A regular hexagon and a regular octagon, both with whole number side lengths, have the same perimeter which is between 80cm and 100cm. What is the number of square centimeters in the area of a square that has the same perimeter as the octagon and the hexagon?

Answers

1. _____
2. _____
3. _____

Solutions to Category 2
Geometry
Meet #2, November 2007

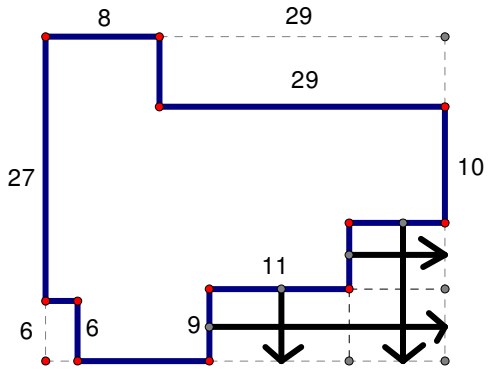
Answers

1. 140

2. 1.5

3. 576

1. The horizontal distance across the top is 8 + 29 = 37, so the 4 horizontal segments across the bottom also have a sum of 37. The vertical distance along the left is 27 + 6 = 33 and the vertical distance along the right will then also be 33. Since the segments can all be moved around to form a rectangle as shown, the perimeter is just 37+33+37+33 = **<u>140</u>**.



2. Since MH and AT are parallel, MATH is a trapezoid and since MH is three times as long as AT, we could call AT =x, and MH =3x. The formula for the area of a Trapezoid is $A_{trap} = 15 = \frac{(b_1 + b_2)h}{2} = \frac{(MH + AT)5}{2} = \frac{(3x + x)5}{2} = \frac{20x}{2} = 10x = 15$

$x = 1.5 = AT$

3. An octagon with whole number side lengths could have perimeter 80, 88, or 96. A hexagon with whole number side lengths could have perimeter 84, 90, 96. So 96 must be the perimeter of both if their perimeters are equal. A square with perimeter 96 would have side lengths of 96÷4 = 24, and an area of 24^2 = **<u>576</u>**.

Category 3
Number Theory
Meet #2, November 2007

1. The prime factorization of 540 is written as $a^x \cdot b^y \cdot c^z$. What is the value of $(a+b+c)-(x+y+z)$?

(note : a, b, c, x, y, and z are not necessarily different)

2. The LCM(x, y) = $54a$ and the GCF(x, y) = $\frac{54}{a}$. What is the value of x · y ?

3. On the 1st day of the year Srinivas, Hector, and Tobias all go to Jo-Jo's pizza shop for dinner. After that day, Srinivas goes to Jo-Jo's every 12th day, Hector goes every 16th day, and Tobias goes every 18th day. On what day of the year will they next all go to Jo-Jo's pizza for dinner? (Note: If the first day of the year is day #1 then the next time they all go to Jo-Jo's is on day # _____)

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #2, November 2007

Answers

1. 4

2. 2916

3. 145

1. The prime factorization of 540 is written as $a^x \cdot b^y \cdot c^z = 2^2 \cdot 3^3 \cdot 5^1$.
The value of
$(a+b+c)-(x+y+z)=(2+3+5)-(2+3+1)=10-6=\underline{\mathbf{4}}$

2. The LCM(x , y) · GCF(x , y) = x · y for all natural numbers x and y.

So LCM(x, y) · GCF(x, y) = $54a \cdot \frac{54}{a} = 54^2 =$ **2916**.

3. Srinivas will go to JoJo's in 12, 24, 36 ..... days.
Hector will go to JoJo's in 16, 32, 48..... days.
Tobias will go to JoJo's in 18, 36, 54..... days.

Since they all go on the multiples of how often they go, we want the LCM(12, 16, 18) = 144. So they will all go again in 144 days, making the next time they all go on the same day the 1^{st} + 144 = Day **145** .

Category 4
Arithmetic
Meet #2, November 2007

1. Bill spends 30% of $\frac{3}{8}$ of the $960 he has in the bank. How much will he have left in the bank after he spends this amount?

2. Simplify $\frac{.5\overline{7}}{.8\overline{6}}$ as a fraction in simplest terms.

3. When the fractions $\frac{7}{12}$ and $\frac{5}{7}$ are added and converted to decimal form, the decimal will have a six digit repetition in it. What is that 6 digit repetition? (note: write your answer as a 6 digit number without decimals or repeating bars. For example, if the decimal was $.978\overline{123456}$ you would write your answer as 123456.)

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #2, November 2007

Answers

1. 852

2. $\frac{2}{3}$

3. 761904
or 619047
or 190476
or 904761
or 047619
or 476190

1. 30% of $\frac{3}{8}$ of \$960 = 30% of $\frac{3}{8}\cdot\frac{960}{1} = 30\%$ of $360 = .3(360) = \$108$

If he spends \$108 he has \$960 – \$108 = **\$852** left over.

2. a) There are several ways to convert repeating decimals into fractions. Once you know the process, this is the fastest :

$$\frac{.5\overline{7}}{.8\overline{6}} = \frac{\frac{57-5}{100-10}}{\frac{86-8}{100-10}} = \frac{\frac{52}{90}}{\frac{78}{90}} = \frac{52}{78} = \frac{4}{6} = \frac{2}{3}$$

b) You could also look at $.5\overline{7}$ as being $5.\overline{7}\div 10 = 5\frac{7}{9}\div 10 = \frac{52}{9}\times\frac{1}{10} = \frac{52}{90}$ and do the same with the other decimal.

c) The more traditional method would be this :

$$\begin{aligned} x &= .5\overline{7} \\ 100x &= 57.\overline{7} \\ -10x &= \ \ 5.\overline{7} \\ 90x &= 52 \\ x &= \frac{52}{90} \end{aligned}$$

3.
$$\begin{array}{r} \frac{7}{12} = .583333333333333333..... \\ +\ \frac{5}{7} = .714285714285714285..... \\ \hline 1.297619047619047619\overline{04} \end{array}$$

You could also add the fractions: $\frac{7}{12}+\frac{5}{7} = \frac{109}{84} = 1\frac{25}{84}$ and then divide 25 by 84.

Category 5
Algebra
Meet #2, November 2007

1. Two years ago, Bob was $\frac{2}{3}$ as old as he will be in 6 years. In how many years from now will Bob be 40 years old?

2. The formula for Volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the formula for ***S***urface ***A***rea of a sphere is $SA = 4\pi r^2$. If the Volume of a given sphere is 972π, what is the Surface Area of the same sphere? Express your answer in terms of π. (note: 972π is an example of a number given "in terms of π".)

3. The sum of seven consecutive multiples of 7 is 1078. What is the sum of the second smallest and the second largest of these seven numbers?

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #2, November 2007

Answers

1. 22

2. 324π

3. 308

1. $b-2=\frac{2}{3}(b+6)$

$b-2=\frac{2}{3}b+4$

$\frac{1}{3}b=6$

$b=18$

So if Bob is 18 years old now, he will be 40 in **<u>22</u>** more years.

2. $972\pi=\frac{4}{3}\pi r^3$

$972=\frac{4}{3}r^3$

$972\cdot\frac{3}{4}=r^3$

$729=r^3$

$9=r$

SO $SA=4\pi 9^2$

$SA=4\pi 81=324\pi$

3. If we call the middle of the seven numbers x, we could use this equation :

$$(x-21)+(x-14)+(x-7)+x+(x+7)+(x+14)+(x+21)=1078$$

$$7x=1078$$

$$x=154$$

Since we want the sum of the 2nd largest and 2nd smallest, we are looking for :

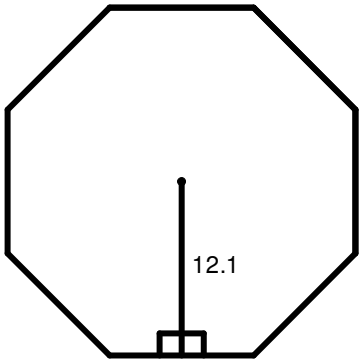
$$(x-14)+(x+14)=2x=2(154)=\underline{\mathbf{308}}$$

Category 6
Team Questions
Meet #2, November 2007

1. What is the positive difference between the sum of the factors of 180 and the sum of the factors of 120?

2. The decimal $.05\overline{108}$ is converted to a fraction in simplest form. What is the numerator of this fraction?

3. A square with area 400 in^2 has the same perimeter as the regular octagon to the right. If the distance from the center of the octagon to the midpoint of one of the sides is 12.1 inches, what is the number of square inches in the area of the octagon?



4. At Tasty Treats candy store they sell candy bars which all cost the same and "penny candy" which each cost one cent. Four friends went into the store. Sally bought 3 candy bars and 8 pieces of penny candy; Joan bought 5 candy bars and 11 pieces of penny candy; Shandra bought 7 candy bars and 4 pieces of penny candy; and Steph bought 4 candy bars and 6 pieces of penny candy. The total bill for the candy was $7.32. How many pieces of penny candy could you buy for the price of 3 candy bars?

5. What is the sum of the five smallest positive numbers each of which have exactly 6 positive factors?

Answers

1. _____ = *A*
2. _____ = *B*
3. _____ = *C*
4. _____ = *D*
5. _____ = *E*
6. _____

6. Using the values the team obtained in questions 1 through 5, find the value of the expression below.

$$\frac{\dfrac{D+E}{13}+\sqrt{C}}{B-A}$$

Solutions to Category 6
Team Questions
Meet #2, November 2007

Answers

1. 186

2. 189

3. 484

4. 111

5. 110

6. 13

1. The sum of the positive integral divisors of 180 =
1+2+3+4+5+6+9+10+12+15+18+20+30+36+45+60+90+180=546

The sum of the factors of 120 =
1+2+3+4+5+6+8+10+12+15+20+24+30+40+60+120 = 360

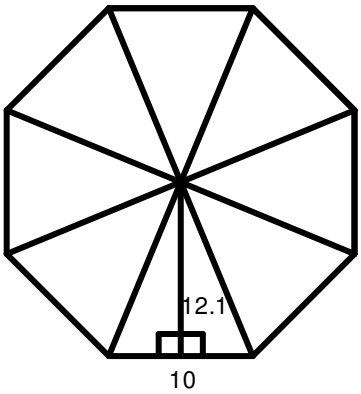
The difference is 546 – 360 =**<u>186</u>**

2. $x = .05\overline{108}$

$100000x = 5108.\overline{108}$

$$\begin{array}{r} -\ 100x = \quad 5.\overline{108} \\ \hline 99900x = 5103 \end{array}$$

$x = \dfrac{5103}{99900} = \dfrac{567}{11100} = \dfrac{189}{3700}$ The numerator is **<u>189</u>**



3. A square with area of 400 has side lengths of $\sqrt{400} = 20$. If the side lengths are 20, then the perimeter is $20 \cdot 4 = 80$. If the perimeter of the square is 80 then the perimeter of the octagon is 80 and each side is $80 \div 8 = 10$. You can divide the octagon up into 8 congruent triangles as shown to the right and each will have an area of $\dfrac{10 \cdot 12.1}{2} = 60.5$, so the area of the octagon is $8 \cdot 60.5 =$ **<u>484</u>**

4. c = the price of a candy bar

$$(3c+8)+(5c+11)+(7c+4)+(4c+6)=732$$
$$19c+29=732$$
$$19c=703$$
$$c=37$$

Three candy bars would cost 3(37) = 111 cents so you could buy **<u>111</u>** pieces of "penny candy".

5. In order for a number to have 6 factors its prime factorization must be in the form $x^1 \cdot y^2$ or x^5. By plugging in small prime numbers for x and y we can find the smallest numbers in those forms.

$3^1 \cdot 2^2 = 12,\ 2^1 \cdot 3^2 = 18,\ 5^1 \cdot 2^2 = 20,\ 7^1 \cdot 2^2 = 28,\ 2^5 = 32$

Any other combination will give a number > 32 so those are the five smallest and the sum is:

12 + 18 + 20 + 28 + 32 = **<u>110</u>**

6. $$\frac{\frac{D+E}{13}+\sqrt{C}}{B-A}=\frac{\frac{110+111}{13}+\sqrt{484}}{189-186}=\frac{\frac{221}{13}+22}{3}=\frac{17+22}{3}=\frac{39}{3}=\mathbf{\underline{13}}$$