Meet #1
October 2007

Intermediate Mathematics League of Eastern Massachusetts

Meet #1
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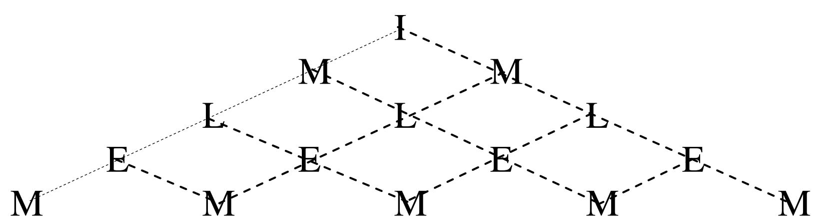
Category 1
Mystery
Meet #1, October 2007

1. How many positive base ten integers less than 10,000 can be written if you only use 1's and 0's?

2. By starting at the I at the top of the diagram below, how many paths could result in spelling IMLEM?



3. Below is a sequence of numbers created by starting with 5 and 7. To get the next terms in the sequence you have to alternate between subtracting the previous two terms and adding the previous two terms. What is the sum of the first 145 terms in this sequence?
(For example first subtract : $5 - 7 = -2$, then add : $7 + (-2)$ to get 5, then go back to subtracting)

$5, 7, -2, 5, -7, -2, \ldots\ldots\ldots\ldots$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1
Mystery
Meet #1, October 2007

Answers

1. 15

2. 16

3. 5

1. There is 1 one digit positive number you could make (1). There are 2 two digit numbers you could make(10, 11). There are 4 three digit numbers you could make(100, 101, 110, 111). There are 8 four digit numbers you could make (1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111). That's a total of **<u>15</u>** positive integers less than 10,000.

You could also say that all the numbers will have 4 digits with a choice of 1 or 0. So there would be $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$ numbers, although 0000 would not be a positive number so there are just **<u>15</u>**.

2. Well, you could try counting them all, but there is an easier way. Starting at the I you have two choices for which M to go to. No matter which M you choose you will have 2 choices for which L to go to. Then you will have 2 choices for which E to go to and finally 2 choices for which M to end on. Therefore there are a total of $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = \underline{\mathbf{16}}$ paths that spell IMLEM.

3. Expanding the sequence we get :

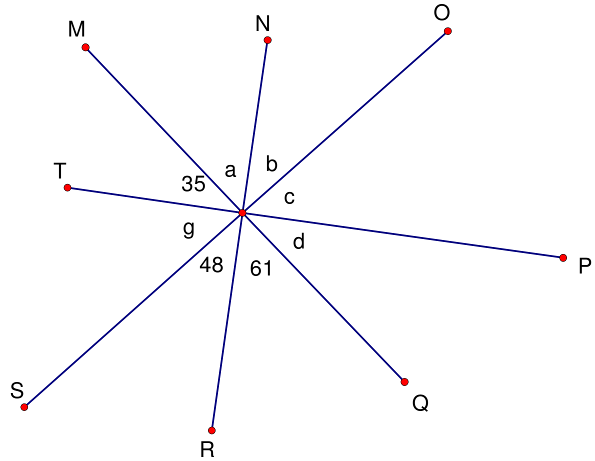
5, 7, – 2, 5, – 7, – 2, – 5, – 7, 2, – 5, 7, 2, 5, 7, – 2, 5, – 7, – 2......

Looking carefully, the first 12 terms are :

5, 7, – 2, 5, – 7, – 2, – 5, – 7, 2, – 5, 7, 2

The sum of these 12 terms is ZERO. After these 12 terms the sequence repeats from the beginning. Since the first 144 terms would include 12 repetitions of the 12 term sequence, the sum of the first 144 terms is also ZERO. Since the 145th term is the same as the first term(5), the sum of the first 145 terms is **<u>5</u>** as well.

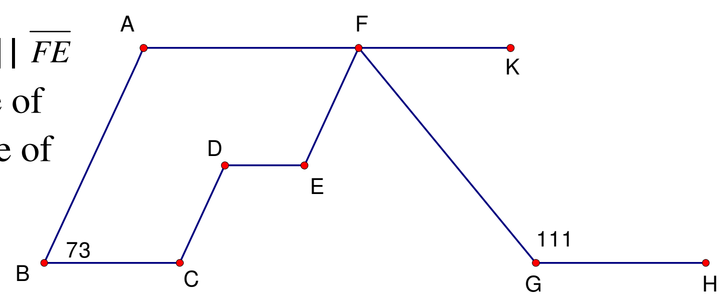
Category 2
Geometry
Meet #1, October 2007



1. In the diagram at the left, the four segments : $\overline{MQ}$, $\overline{NR}$, $\overline{OS}$, and $\overline{PT}$ all intersect at the same point creating 8 angles labeled with either their angle measures or a variable name. What is the value of a + b + c + d + g?

2. Two angles have a sum of 170 degrees. The complement of one of the angles is equal to the supplement of the other. How many degrees are in the measure of the larger angle ?

3. In the diagram to the right, $\overline{AB} \parallel \overline{DC} \parallel \overline{FE}$ and $\overline{BC} \parallel \overline{DE} \parallel \overline{GH} \parallel \overline{AK}$. The measure of angle ABC is 73 degrees and the measure of angle FGH is 111 degrees. What is the measure of angle EFG?
(the symbol || means "is parallel to")



Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 2
Geometry
Meet #1, October 2007

Answers

1. 216

2. 130

3. 38

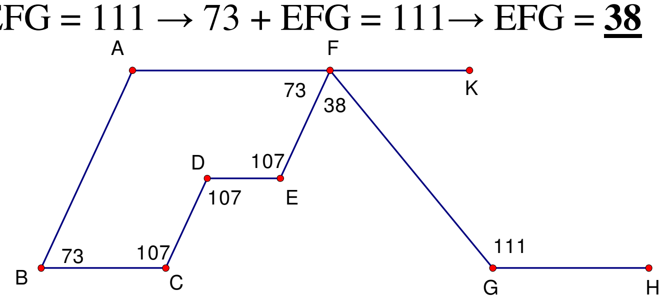
1. Since there is a total of 360 degrees around a point, all the angles in the diagram must add up to 360 degrees.

So $a + b + c + d + g + 35 + 48 + 61 = 360 \rightarrow g + 144 = 360 \rightarrow g = 216$.
Therefore the sum of a, b, c, d & g = **<u>216</u>** degrees.

2. The complement of an angle x is $90 - x$ and the supplement of an angle y is $180 - y$. If the complement of one equals the supplement of the other $90 - x = 180 - y \rightarrow y - x = 90$. That means the difference of the two angles is 90 degrees. Since we know the sum is 170 and one of them is 90 more than the other, simple algebra or guess and check can find that 130 & 40 fit both those requirements and **<u>130</u>** is the larger number.

3. Since AB and DC are parallel, the same side interior angles(ABC and BCD) have a sum of 180 degrees. So angle BCD has a measure of 107 degrees. Angle BCD is alternate interior to EDC which is alternate interior to DEF, so angle BCD has the same measure as DEF(107 degrees). Angle AFE and DEF are same side interior angles they have a sum of 180 degrees and angle AFE has a measure of 73. Since AFG is alternate interior with FGH, AFG has a measure of 111 degrees. Since angles AFE and EFG form angle AFG, their sum is 111 degrees. So :

AFE + EFG = 111 $\rightarrow$ 73 + EFG = 111 $\rightarrow$ EFG = **<u>38</u>**



Category 3
Number Theory
Meet #1, October 2007

1. Billy says the divisibility rule for 60 is that you check and see if the number is divisible by 6 and 10. Sally says that he is wrong(and Sally is correct). Sally says that the best way to check if a number is divisible by 60 is to see if it is divisible by X, Y, and Z with X, Y, and Z all greater than 1. What is the minimum value of $Y \cdot (X+Z)$?

2. If P represents the sum of the prime numbers between 40 and 60, and C represents the sum of the odd composite numbers between 40 and 60, what is the positive difference between P and C?

3. How many factors of 432 are the squares of positive integers?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #1, October 2007

Answers

1. 27

2. 14

3. 6

1. Billy is wrong because numbers like 30 are divisible by 6 and 10, but not by 60. Since 60 is equal to $3 \cdot 4 \cdot 5$ and 3, 4, and 5 are relatively prime, any number divisible by 60 must be divisible by 3, 4, and 5. Therefore X, Y, and Z are equal to 3, 4, and 5 in some order. Checking the three possible combinations we can find that the minimum value of $Y \cdot (X+Z)$ is then $3(4 + 5) = 3(9) =$ **<u>27</u>**.

2.

$$P = 41 + 43 + 47 + 53 + 59 = 243$$
$$C = 45 + 49 + 51 + 55 + 57 = 257$$

$$C - P = 257 - 243 = \underline{\mathbf{14}}$$

3. You could list out all the factors and pick out the squares :

<u>1</u>, 2, 3, **<u>4</u>**, 6, 8, **<u>9</u>**, 12, **<u>16</u>**, 18, 24, 27, **<u>36</u>**, 48, 54, 72, 108, **<u>144</u>**, 216, 432

So there are 6 factors of 432 that are perfect squares.

For a more elegant solution

You could look at the ways a perfect square might be made from factors by looking at the prime factorization.(*note : prime factorization is not necessary for this problem as shown first, this is just an alternate solution)

$$432 = 2^4 \text{ x } 3^3$$

Since square numbers have an even number of the same factors multiplied together, a perfect square factor of 432 could be divisible by 2^0, 2^2, or 2^4. It could also be divisible by 3^0 or 3^2. That gives us 3 choices for divisibility by two and 2 choices for divisibility by three. That gives us a total of $3 \text{x} 2 =$ **<u>6</u>** square factors of 432.

Category 4
Arithmetic
Meet #1, October 2007

1. Find the value of this expression : $\dfrac{2^3 \cdot 3^2 + 2^2(1 + 2 \cdot 3)}{3^2 + 4^2}$.

2. The stem and leaf plot below shows the number of hits each player for the Boston Red Sox had at the end of the season.

What is the mean number of hits for the Red Sox?

Red Sox

20	3
19	2 5
18	2 4 6
17	6
16	3
15	7

3. In a list of 5 positive integers, the mean is 17, the median is 18, and the mode is unique. What is the largest possible number in the list?
**"The mode is unique" means that there is just one mode. **

Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 4
Arithmetic
Meet #1, October 2007

Answers

1. 4

2. 182

3. 46

1.

$$\frac{2^3 \cdot 3^2 + 2^2(1+2\cdot 3)}{3^2+4^2} = \frac{2^3 \cdot 3^2 + 2^2(7)}{3^2+4^2}$$

$$= \frac{8\cdot 9 + 4(7)}{9+16} = \frac{72+28}{9+16} = \frac{100}{25} =$$

$$\underline{\mathbf{4}}$$

2. The Red Sox mean = (203+195+192+186+184+182+176+163+157) $\div$9 = **<u>182</u>**

** You can add numbers in a stem and leaf plot without actually adding them!!! You can add the leaves →3+5+2+6+4+2+6+3+7=38 and add the stems(once each per leaf) →20+19+19+18+18+18+17+16+15=160 separately. Since the stems were actually the tens place, multiply that sum by 10 : 160x10 = 1600 and then add the leafs which came from the ones place 1600+38 = 1638. Now you can divide by 9 to get 182!

3. In a list of 5 positive integers, the mean is 17, the median is 18, and the mode is unique. What is the largest possible number in the list?

If the mean is 17, then the sum of the numbers must be 5x17 = 85. The median is 18, so the middle number will be 18. If we want the biggest possible number we want the rest to be small, so we should make the first two as small as possible(both 1). The 4th number must be greater than 18 since if it were 18 there would be two modes(1 & 18), so the 4th number is 19. Adding up the 4 we have so far gives us 39. The last number would then be 85 – 39 = **<u>46</u>**. If you make the smaller two numbers 1 and 2 you can make the 4th number 18, but those first four numbers would still have a sum of 39 and the biggest number would still be **<u>46</u>**.

Category 5
Algebra
Meet #1, October 2007

1. Find the value of M so that the equation below is an identity.

$$3(2x + 4M) + 16 + 5x = 8(3x - 4) - 13x$$

2. Evaluate the expression below for x = 4 and $y = -\frac{1}{2}$.

$$\left(2x + 8y\right) - \left(6y - \frac{x}{2}\right)$$

3. Solve for x :

$$3x + 7 + 2\left(8 + 4x\right) = 5\left(2x + 9\right) - x$$

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #1, October 2007

Answers

1. – 4

2. 9

3. 11

1.

$$3(2x+4M)+16+5x=8(3x-4)-13x$$
$$6x+12M+16+5x=24x-32-13x$$
$$11x+12M+16=11x-32$$
$$12M=-48$$
$$M=\underline{\mathbf{-4}}$$

2.

$$(2x+8y)-\left(6y-\frac{x}{2}\right)=$$
$$\left(2(4)+8\left(-\frac{1}{2}\right)\right)-\left(6\left(-\frac{1}{2}\right)-\frac{4}{2}\right)=$$
$$(8-4)-(-3-2)=$$
$$4-(-5)=$$

$$\underline{\mathbf{9}}$$

3.

$$3x+7+2(8+4x)=5(2x+9)-x$$
$$3x+7+16+8x=10x+45-x$$
$$11x+23=9x+45$$
$$2x=22$$
$$x=\underline{\mathbf{11}}$$

Category 6
Team Questions
Meet #1, October 2007

1. Two positive integers (x, y), with $x \le y$, have an mean of 40. What is the median of the non-negative differences between y and x for each such pair of numbers?
(note : (36,44) has a mean of 40 and a positive difference of 8)

2. The functions $\varnothing$ and $\otimes$ and $\boxdot$ are defined as follows :

$$a \varnothing b = ab - a - b \; ; \; a \otimes b = a(a+b) \; ; \; a \boxdot b = b(a+b)$$

What is the value of $2 \varnothing [3 \otimes (4 \boxdot 5)]$?

3. How many numbers between 100 and 150 are the product of two distinct prime numbers?
(NOTE: while 12 is the product of 2's and 3's, it is NOT the product of two primes since it is 2x2x3, the product of three primes)

4. What is the third smallest positive integer that leaves a remainder of 2 when divided by 3, when divided by 11, and when divided by 19? (Count the number 2 as the smallest positive integer that leaves a remainder of 2 when divided by each of 3, 11 and 19.)

5. On a recent test, the class mean of the 40 students was 86%. If the boys in the class averaged 80% and the girls averaged 90%, how many more girls than boys are in the class?

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. ______________

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below.

$$C \div \sqrt{\left(\sqrt{D - B \cdot E - A}\right)}$$

Solutions to Category 6
Team Questions
Meet #1, October 2007

Answers

1. 39

2. 142

3. 15

4. 1256

5. 8

6. 5

1. The possible pairs (x, y) that have a mean of 40 with $x \le y$ are :
(1,79) ;(2, 78); (3, 77); (4, 76) ....(40, 40)
The differences in these pairs are :
78, 76, 74, 72, ..., 6, 4, 2, 0

a)You can find the median by listing them all and finding 39 is between 38 and 40.

b)You could also remove sets of numbers from either end. Taking away 78 thru 70 and 8 thru 0, then 68 thru 60 and 18 thru 10. Continuing that process you'd last remove 40 and 38 giving 39 the median

c)By noticing that (78+0)/2 or (76+2)/2 or (74+4)/2 are all the same, you could notice that 39 is always in the middle(median).

The median is <u>**39**</u>

2. Working from the inner parenthesis of $2 \varnothing [3 \otimes (4 \boxdot 5)]$ and out :

$4 \boxdot 5 = 5(4+5) = 5(9) = 45$, giving us $2 \varnothing (3 \otimes 45)$
$3 \otimes 45 = 3(3+45) = 3(48) = 144$, giving us $2 \varnothing 144$
$2 \varnothing 144 = 2(144) - 2 - 144 =$ <u>**142**</u>.

3. By using an organized list starting with the smallest prime :
2x53, 2x59, 2x61, 2x67, 2x71, 2x73
3x37, 3x41, 3x43, 3x47
5x23, 5x29
7x17, 7x19
11x13 (can't count 11x11 as those are not two DISTINCT primes)

Total of <u>**15**</u> numbers between 100 and 150 without even finding the 15 numbers!!!

4. If the remainder is always 2, the number must be 2 more than a multiple of 3, 11, and 19. Multiplying 3, 11, and 19 together gives a product of 627. Adding 2 gives the 2^{nd} smallest integer with the correct remainders. Multiplying 627 by 2 and then adding 2 gives us **<u>1256</u>** which is 2rd smallest integer which gives a remainder of 2 when divided by 3, 11, or 19.

Note: The original problem asked for the "third smallest integer that leaves a remainder of 2 when divided by 3, 11 or 19," which some interpreted as meaning you could just consider 3 as a divisor, so the answer would be 8. The question was changed for the archive due to this. Thanks to the Carlisle team for pointing this out 11/1/2007.

5. There are several ways to do this. One is to notice that the overall average is six tenths of the way between 80 and 90, so six tenths of the people must have scored 90%. Six tenths of 40 equals 24 girls and therefore 16 boys.

Another way is to write the equation that would be used to calculate average. Using b and the number of boys and g as the number of girls we have :

$\frac{80b + 90g}{40} = 86$ Simplifying this we get : $8b + 9g = 344$

Since we know b + g = 40, 8b + 8g = 320. You can see the extra g in the simplified equation added 24 to the right, so there must be 24 girls and then 16 boys. So there are **<u>8</u>** more girls then.

6.

$$C \div \sqrt{\left(\sqrt{D - B \cdot E - A}\right)}$$

$$15 \div \sqrt{\left(\sqrt{1256 - 142 \cdot 8 - 39}\right)}$$

$$15 \div \sqrt{\left(\sqrt{1256 - 1136 - 39}\right)}$$

$$15 \div \sqrt{\left(\sqrt{81}\right)} = 15 \div \sqrt{9} = 15 \div 3 =$$

<u>5</u>