Meet #5
March 2006

Intermediate Mathematics League of Eastern Massachusetts

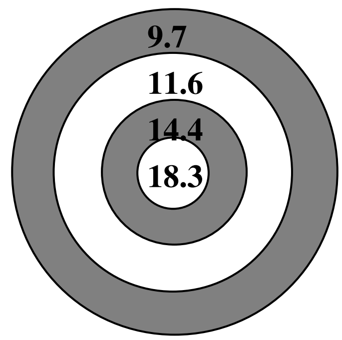
Meet #5
March 2006

Category 1
Mystery
Meet #5, March 2006

You may use a calculator today.

1. The combined cost of a movie ticket and popcorn is $8.00. If the ticket costs $5 more than the popcorn, how much is the popcorn? Give your answer in dollars to the nearest hundredth of a dollar.

2. Isabella tossed three darts at a dart board that looked like the one shown here. If all three darts stuck somewhere on the board and her total score was a whole number less than 40, what was her total score?



3. A digital clock displays the hour and the minutes of the time of day. From midnight to noon, how many more minutes have at least one 4 appearing than minutes with at least one 7 appearing on a digital clock?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 1
Mystery
Meet #5, March 2006

Answers

1. $1.50

2. 31

3. 99

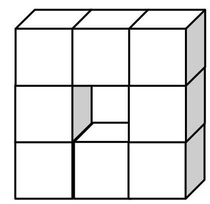
1. One strategy for this kind of problem is to "subtract to a tie." Suppose you have a coupon for $5.00 off the price of the movie ticket. Now the ticket is the same price as the popcorn, and you would pay a total of $8 – $5 = $3. Dividing $3 by 2, we find that the popcorn must cost **$1.50**. Incidently, the movie ticket must cost $1.50 + $5 = $6.50.

2. There are two ways to get a whole number score with the point values given on the dart board. One way is to get 18.3 + 18.3 + 11.6, which gives a total score of 48 points. The other way is to get 9.7 + 9.7 + 11.6, which gives a total score of 31 points. Since Isabella's total score was less than 40 points, she must have had a total score of **31** points.

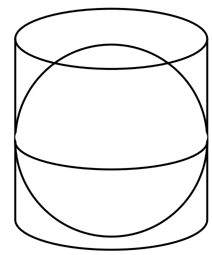
3. From midnight to noon, there will be at least one 4 from 4:00 to 4:59 and at least one 7 from 7:00 to 7:59. Likewise in every set of 10 minutes, a 4 occurs in the ones place as does a 7. The extra 4's come from the fact that 4's will appear in the tens place, whereas a 7 will never appear in the tens place. We can only count 9 extra minutes in the 40's, however, since we already counted 44 as having *at least one 4*. Thus there are 9 extra minutes with at least one 4 for each of the 11 hours besides the 4:00 to 4:59 hour. That's 9 times 11, which is **99**.

Category 2
Geometry
Meet #5, March 2006

You may use a calculator today.

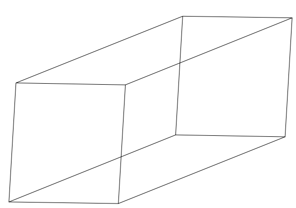


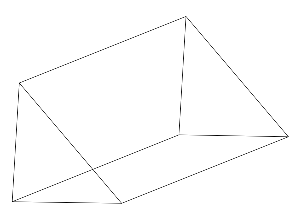
1. The figure at right is made from 8 unit cubes that are glued together. There is a hole through the middle of the object. How many square units are in the surface area of the entire figure?



2. A plastic sphere with an radius of 10 cm is full of water. If the water from this sphere is poured into a cylinder that has a radius of 10 cm and a height of 20 cm, what fraction of the cylinder will be filled with water? Disregard the thickness of the plastic and express your answer as a simplified fraction.

3. The base of the prism on the left is a square with a side length of 1 unit. The base of the prism on the right is a right triangle with legs of length 1 unit. Both prisms are 2 units long. What is the difference between the surface area of the square based prism on the left than the surface area of the triangular based prism on the right? Express your answer as a decimal to the nearest thousandth of a square unit. (Note that what is called the base may not be on the bottom.)





Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 2
Geometry
Meet #5, March 2006

Answers

1. 32

2. $\frac{2}{3}$

3. 2.172

1. There are 8 unit squares on the front and 8 unit squares on the back of the figure. There are 12 unit squares around the outer rim and 4 unit squares around the inner rim. The total is thus 8 + 8 + 12 + 4 = **32** unit squares.

2. The particular radius of the sphere doesn't matter, as long as the height of the cylinder is equal to two radii, which is the diameter of the sphere. In genereal, the volume of the sphere is $\frac{4}{3}\pi r^3$, and the volume of the cylinder is $\pi r^2 h = \pi r^2 2r = 2\pi r^3$. The ratio of their volumes is thus $\frac{\frac{4}{3}\pi r^3}{2\pi r^3} = \frac{\frac{4}{3}}{2} = \frac{4}{3} \div 2 = \frac{2}{3}$, so the cylinder will be $\mathbf{\frac{2}{3}}$ full of water.

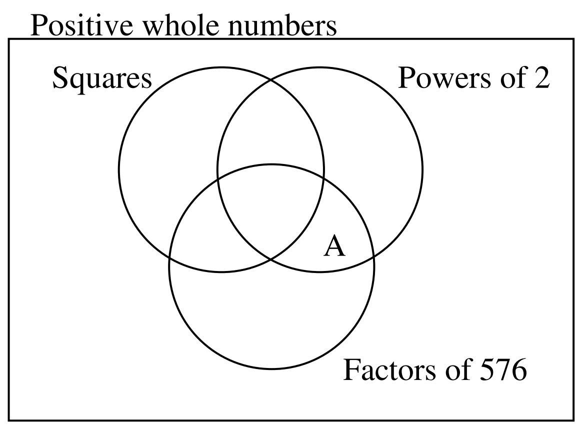
3. The square based prism on the left has six faces: two that are unit squares and four that are 1 by 2 rectangles. The surface area of the square based prism is thus $2 \times 1 \times 1 + 4 \times 1 \times 2 = 10$ square units. The triangular based prism on the right has five faces: two that are isosceles right triangles with legs of 1 unit, two that are 1 by 2 rectangles, and one that is a rectangle with a width of $\sqrt{2}$ and a length of 2. The surface area of the triangular prism is $2 \times \frac{1}{2} \times 1 \times 1 + 2 \times 1 \times 2 + 1 \times \sqrt{2} \times 2$ $= 5 + 2\sqrt{2} \approx 7.828$ square units. The desired difference is 10 – 7.828 = **2.172**.

Category 3
Number Theory
Meet #5, March 2006

You may use a calculator today.

1. In a class of 23 students, there are 19 mouths that speak but only 11 heads that think. If there is one student in the class who neither speaks nor thinks, how many students think but don't speak?

2. The rectangle at right represents the "universe of discourse." That's a fancy way of saying "everything we are talking about," which in this case is positive whole numbers. One circle contains the perfect square numbers, one circle contains the powers of two, and one circle contains the factors of 576. Find the sum of all the numbers that belong in region A.



3. Let the 500 students at De Morgan's Middle School be our universe of discourse. Let M be the set of the 120 students who compete in math contests and let C be the set of the 150 students who sing in chorus. There are 70 students who participate in both of these activities. The complement of a set is the set of all the elements that are not in the set but are still in the universe of discourse. How many students are in the complement of $M \cup C$? In set notation, we are looking for $\left|\overline{(M \cup C)}\right|$, where the horizontal bar denotes the complement and the vertical bars indicate the size of the set.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3
Number Theory
Meet #5, March 2006

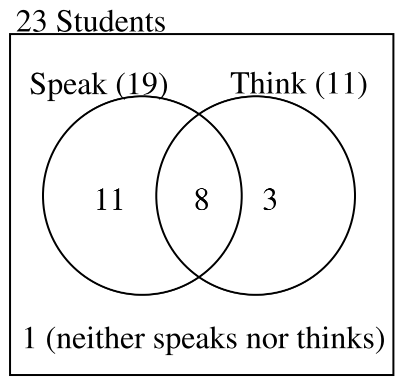
Answers

1. 3

2. 42

3. 300

1. There are 23 students in the class, but one student neither speaks nor thinks, so there are 22 students who either speak or think or both. Since 19 + 11 = 30 and 30 – 22 = 8, there must be 8 students who speak and think. This means there are 11 – 8 = **3** students who think but don't speak.



2. We are looking for the powers of two that are factors of 576 and are *not* square numbers. The powers of two are: 1, 2, 4, 8, 16, 32, 64, etc. Every other power of two is a square number, namely: 1, 4, 16, 64, etc. The powers of two that are *not* square numbers are 2, 8, 32, 128, etc. We need to find the largest power of 2 that is a factor of 576. It turns out that 576 = 64 x 9. Thus the sum we are looking for is 2 + 8 + 32 = **42**.

3. If you add the 120 students who compete in math contests and the 150 students who sing in chorus, you are double counting the 70 people who do both activities. Thus, there are 120 + 150 – 70 = 200 students at De Morgan's Middle School who participate in at least one of these two groups. In other words, $|M \cup C| = 200$. There must be 500 – 200 = 300 students who do *neither* activity, which means $\left|\overline{(M \cup C)}\right| = 300$.

Category 4
Arithmetic
Meet #5, March 2006

You may use a calculator today.

1. For his daughter's birthday, Thornton plans to buy one of five different models of bicycle, one of four different styles of helmet, and either a bell or a horn. How many different set-ups of bicycle, helmet, and noise maker are possible?

2. There were 8 people stranded on a desert island when a lifeboat washed up on the beach. Unfortunately, the lifeboat would only hold 4 people. How many different groups of 4 people could be chosen from the 8 people to go off in the lifeboat?

3. In the game of Backgammon, there is a cube with the following powers of two on its six faces: 2, 4, 8, 16, 32, and 64. Consider the possible sums you get when this cube and a number cube with whole numbers 1 through 6 are rolled together. What is the probability that the sum of the numbers on the tops of the two cubes is a prime number? Express your answer as a common fraction in lowest terms.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4
Arithmetic
Meet #5, March 2006

Answers

1. 40

2. 70

3. $\frac{11}{36}$

1. For each of the 5 different bicycles Thornton could buy, there are 4 helmet, and 2 noise makers. We use the multiplication principle to calculate that there are $5 \times 4 \times 2 = \mathbf{40}$ different possible set-ups.

2. This is a "combination" problem, rather than a "permutation" problem, since the order in which the four people are chosen does not matter. If it were a permutation problem, we would simply multiply as follows $8 \times 7 \times 6 \times 5 = 1680$. But the same four people could be chosen in any of $4 \times 3 \times 2 \times 1 = 24$ different ways, so we must divide 1680 by 24 to get **70** different groups of 4 people. Using the general formula for combinations, we calculate as follows:

$$_{12}C_4 = \frac{8!}{(8-4)! \cdot 4!} = \frac{8!}{4! \cdot 4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5 = 70.$$

3. The 36 possible sums are shown in the table below with the 11 prime numbers in bold. The probability of rolling a prime sum is $\mathbf{\frac{11}{36}}$.

+	2	4	8	16	32	64
1	**3**	**5**	9	**17**	33	65
2	4	6	10	18	34	66
3	**5**	**7**	**11**	**19**	35	**67**
4	6	8	12	20	36	68
5	**7**	9	**13**	21	**37**	69
6	8	10	14	22	38	70

Category 5
Algebra
Meet #5, March 2006

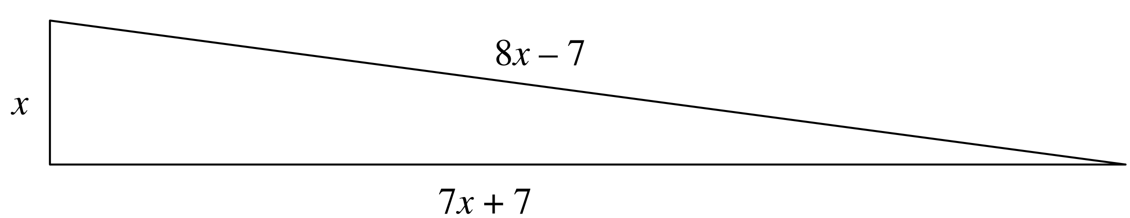
You may use a calculator today.

1. Find the negative value of x that makes the following equation true:

$$x(x+1)=210$$

2. A rational number and its reciprocal have a sum of $2\frac{9}{10}$. If the number is less than one, what is this number? Express your answer as a common fraction in simplest form.

3. The triangle below is a right triangle with side lengths given in terms of x. How many square units are in the area of the triangle? Express your answer to the nearest whole number of square units.



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #5, March 2006

Answers

1. -15

2. $\frac{2}{5}$

3. 840

1. Rather than set the quadratic equation equal to zero, consider what the equation says in its current form. The product of two numbers is equal to 210. Since the two numbers are one apart, the square root of 210 will put us close to the correct values. If x were positive, it would be 14 times 15 equals 210. Since x is negative, it's –15 times –14. So $x =$ **–15** is it.

2. The English translates to the equation $x + \frac{1}{x} = 2\frac{9}{10}$, or $x + \frac{1}{x} = \frac{29}{10}$. Multiplying both sides of the equation by $10x$, we get $10x^2 + 10 = 29x$. We now subtract $29x$ from both sides to get $10x^2 - 29x + 10 = 0$. Now we can either use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, or we can try to factor this trinomial into the product of two binomials. Factoring gives us the equation $(5x-2)(2x-5) = 0$. If $5x - 2 = 0$, then the solution is $x = \frac{2}{5}$. If $2x - 5 = 0$, then the solution is $x = \frac{5}{2}$. We want the solution that is less than 1, which is $\mathbf{\frac{2}{5}}$.

3. By the Pythagorean Theorem, we write the equation $x^2 + (7x+7)^2 = (8x-7)^2$. Expanding on both sides, we get $x^2 + 49x^2 + 98x + 49 = 64x^2 - 112x + 49$. Subtracting 49 from both sides and combining like terms, we get $50x^2 + 98x = 64x^2 - 112x$. We can now set this equation equal to zero by subtracting $50x^2$ and $98x$ from both sides. This gives us the equation $0 = 14x^2 - 210x$. Factoring out $14x$ from both terms, we rewrite the equation to get $0 = 14x(x-15)$. The solutions are $x = 0$, which is not useful to us, or $x = 15$, which is useful to us. The height of the triangle is 15 units and the base is $7 \times 15 + 7 = 105 + 7 = 112$ units. The area of the triangle is thus $A = \frac{1}{2} \times 112 \times 15 = 56 \times 15 = \mathbf{840}$ square units.

Category 6
Team Questions
Meet #5, March 2006

You may use a calculator today.

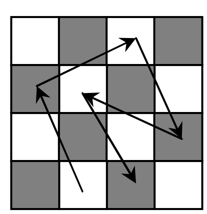
1. Peter has an average of 85% on 5 tests. If 100% is the maximum score, all scores are whole number percentages, and all five scores are different, what is the least possible score that Peter could have on one of his tests?

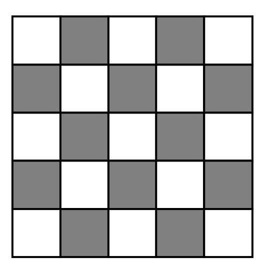
2. In the game of Backgammon, there is a cube with the following powers of two on its six faces: 2, 4, 8, 16, 32, and 64. If this cube and a standard pair of dice are all rolled together, what is the probability that the sum of the numbers on the tops of the three cubes is a prime number? Express your answer as a common fraction in lowest terms.

3. Jar A contains 2 red marbles and 1 blue marble. Jar B contains 1 red and two blue marbles. A marble is selected at random from jar A and placed with the other marbles in jar B. Then a marble is selected from jar B. What is the probability that the marble selected from jar B is red?

4. A triangular number is a number that is the sum of consecutive integers starting with 1. A "triangle triple" is a set of three triangular numbers where the sum of the lesser two is equal to the third. How many triangle triples are there where all three triangular numbers are less than 70? Note: A triangle triple may use the same number twice, but the order of the sum does not matter.

5. A knight can make 5 moves on a 4-by-4 chessboard without crossing its own path, as shown in the picture below left. How many moves can a knight take on a 5-by-5 chessboard without crossing its own path? You can start the knight anywhere you like.





6. Using the values the team obtained in questions 1 through 5, you should get an integer answer when you evaluate the following expression:

$$(A - D + E)(B + C)$$

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. _________________

Solutions to Category 6
Team Questions
Meet #5, March 2006

Answers

1. 31

2. $\frac{11}{36}$

3. $\frac{5}{12}$

4. 5

5. 10

6. 26

1. If Peter has an average of 85% on his 5 tests, then the sum of those five scores is 85 $85 \times 5 = 425$. If one of the scores is to be the lowest possible, then the other four scores must be the highest possible without being the same. Since 100 + 99 + 98 + 97 = 394, the lowest score could have been 425 – 394 = **31**.

2. There are 216 different ways the three number cubes can land. The difficulty here is finding a way to organize all the sums. If we consider the sums in the table we made for Arithmetic problem #3, we need only add 2, 4, and 6 to the odd rows and 1, 3, and 5 to the even rows. The six half-tables below give all 108 possible *odd* sums with primes in bold.

+	2	4	8	16	32	64
1	**3**	**5**	9	**17**	33	65
2	4	6	10	18	34	66
3	**5**	**7**	**11**	**19**	35	**67**
4	6	8	12	20	36	68
5	**7**	9	**13**	21	**37**	69
6	8	10	14	22	38	70

+	3	5	9	17	33	65
2	**5**	**7**	**11**	**19**	35	**67**
4	**7**	9	**13**	21	**37**	69
6	9	**11**	15	**23**	39	**71**

+	4	6	10	18	34	66
1	**5**	**7**	**11**	**19**	35	**67**
3	**7**	9	**13**	21	**37**	69
5	9	**11**	15	**23**	39	**71**

+	5	7	11	19	35	67
2	**7**	9	**13**	21	**37**	69
4	9	**11**	15	**23**	39	**71**
6	**11**	**13**	**17**	25	**41**	**73**

+	6	8	12	20	36	68
1	**7**	9	**13**	21	**37**	69
3	9	**11**	15	**23**	39	**71**
5	**11**	**13**	**17**	25	**41**	**73**

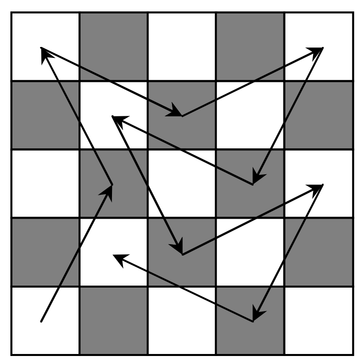
+	7	9	13	21	37	69
2	9	**11**	15	**23**	39	**71**
4	**11**	**13**	**17**	25	**41**	**73**
6	**13**	15	**19**	27	**43**	75

+	8	10	14	22	38	70
1	9	**11**	15	**23**	39	**71**
3	**11**	**13**	**17**	25	**41**	**73**
5	**13**	15	**19**	27	**43**	75

There are 11 primes in each of the six half-tables above. That's $6 \times 11 = 66$ primes out of 216 possible sums. The probability of getting a prime sum when these three cubes are rolled is thus $\frac{66}{216} = \mathbf{\frac{11}{36}}$.

3. We have to consider two possible scenarios. (1) There is a $\frac{2}{3}$ chance that a red marble is chosen from jar A and placed in jar B. Then there would be a $\frac{1}{2}$ chance of choosing a red marble from jar B. The probability of scenario 1 is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$. (2) There is a $\frac{1}{3}$ chance that a blue marble is chosen from jar A and placed in jar B. Then there would be a $\frac{1}{4}$ chance that a red marble is chosen. The probability of scenario 2 is $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$. The combined probability of choosing a red marble from jar B is thus $\frac{1}{3} + \frac{1}{12} = \frac{4}{12} + \frac{1}{12} = \mathbf{\frac{5}{12}}$. Note that the probability of choosing a blue marble is $\frac{1}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{2}{4} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$.

4. The five (**5**) triangle triples with all three numbers less than 70 are: 3 + 3 = 6, 6 + 15 = 21, 10 + 45 = 55, 15 + 21 = 36, and 21 + 45 = 66.



5. The knight can make ten (**10**) moves without crossing its own path. One possible solution is shown at right.

6. Substituting the correct values into the expression, we get:

$$\begin{aligned}(A - D + E)(B + C) &= (31 - 5 + 10)\left(\frac{11}{36} + \frac{5}{12}\right) \\ &= 36\left(\frac{11}{36} + \frac{15}{36}\right) \\ &= 36\left(\frac{26}{36}\right) \\ &= \mathbf{26}\end{aligned}$$