Meet #4
February 2006

Intermediate Mathematics League of Eastern Massachusetts

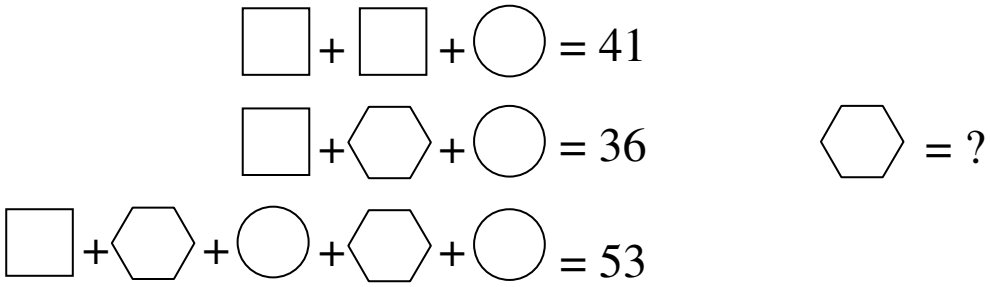
Meet #4
February 2006

You may use a calculator

Category 1
Mystery
Meet #4, February 2006

1. Last month, a company spent $74,000 on 37-cent stamps. If they send out the same number of letters this month, how many dollars can they expect to spend on first class stamps now that stamps cost 39 cents? Round your answer to the nearest thousand dollars.

2. In the picture below, same shapes have the same number. What number must go in the hexagons to make all the equations true?



3. Abby, Billy, and Casey were outside playing in the snow. They each wore a hat and a scarf. They decided to trade hats and scarves so that none of them would have their own hat or scarf and nobody would have both items from the same person. How many different ways can they do this?

Answers

1. _____
2. _____
3. _____

Solutions to Category 1
Mystery
Meet #4, February 2006

Answers

1. $78,000

2. 14

3. 2

1. If the company spent $74,000 on 37-cent stamps, they must have mailed 74,000 ÷ 0.37 = 200,000 letters. If they mail the same number of letters this month, then it will cost 200,000 × 0.39 = **$78,000**.

2. The third equation has all the shapes that the second equation has, plus an extra hexagon and an extra circle. These two together account for the extra 53 – 36 = 17 units. Taking this value back to the second equation, we can determine that the square must be 36 – 17 = 19 units. The two squares in the first equation account for 2 × 19 = 38 units, so the circle must be 41 – 38 = 3 units. Finally, the hexagon and the circle are 17 units, so the hexagon must be 17 – 3 = **14** units.

3. From Abby's point of view, there are only two choices for which hat she can wear, Billy's or Casey's. Once she has chosen a hat, then everything else is forced. Thus there are only two (**2**) ways they can trade hats and scarves so that none of them would have their own hat or scarf and nobody would have both items from the same person.

To verify, let's make groups of three letters to keep track of who's wearing what. The first letter will be the first initial of the name of the person, the second letter will be for the owner of the hat, and the third letter for the scarf. When we count "different ways they can do this," we must consider all three kids for each "way."

One way is:	ABC	BCA	CAB
Another way is:	ACB	BAC	CBA

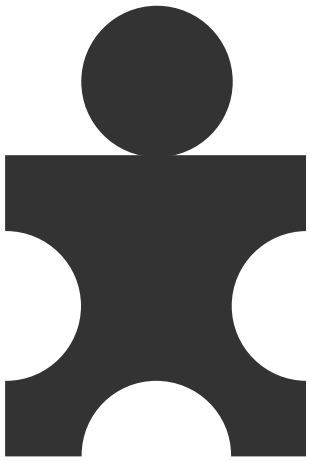
Category 2
Geometry
Meet #4, February 2006

You may use a calculator

1. A toy car has wheels with a diameter of 1 inch. How many turns does each wheel make if the car rolls 12 feet across the floor? (There are 12 inches in 1 foot.) Use 3.14 for π and round your answer to the nearest whole number of turns.

2. How many centimeters are there in the circumference of a circle with an area of 36π square centimeters? Use 3.14 for π and express your answer as a decimal to the nearest tenth of a centimeter.

3. Three semi-circles of diameter 2 centimeters are cut from three sides of a 4-cm by 4-cm square to form the figure below. A circle of radius 1-cm is placed above the square without overlap. How many square centimeters are in the area of the figure? Use 3.14 for π and express your answer to the nearest tenth of a square centimeter.



Answers

1. _____
2. _____
3. _____

Solutions to Category 2
Geometry
Meet #4, February 2006

Answers

1. 46

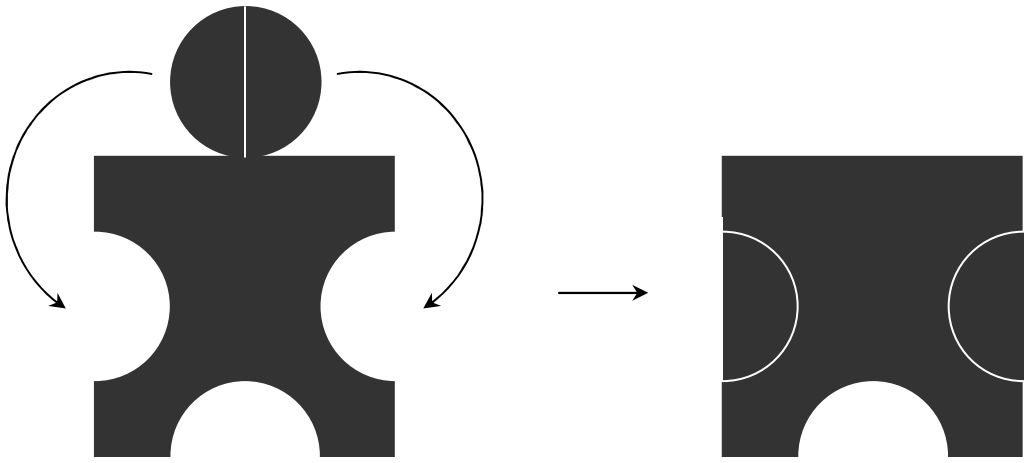
2. 37.7

3. 14.4

1. Wheels with a diameter of 1 inch have a circumference of $\pi \times 1 = \pi$ inches, which is about 3.14 inches. If the car rolls 12 feet across the floor, then it rolls $12 \times 12 = 144$ inches. The question now is how many turns of 3.14 inches there are in 144 inches. Dividing 144 by 3.14, we get about 45.86 turns, which is **46** to the nearest whole number of turns.

2. The formula for the area of a circle is $A_{\text{circle}} = \pi r^2$. We can find the radius of the given circle by solving the equation $36\pi = \pi r^2$. Since $6^2 = 36$, the radius must be 6 centimeters. The formula for the circumference of a circle is $C = \pi D$ or $2\pi r$, so the circumference of our circle is 12π. Using 3.14 as an approximation of π, we get $12 \times 3.14 = 37.68$ or **37.7** to the nearest tenth of a centimeter.

3. If we cut the circle above the square in half, we can fill two of the voids on the sides of the square That would leave a square with just one semicircular region cut out of it. The area of the square is 4 cm × 4 cm = 16 square centimeters. The area of a *circle* with radius 1 centimeter is $\pi \times 1^2 = \pi$ square centimeters, so the area of a *semicircle* with radius 1 cm is 0.5π. Thus the figure has an area of $16 - 0.5\pi = 16 - 1.57 = 14.43$ or **14.4** square centimeters to the nearest tenth.



Category 3
Number Theory
Meet #4, February 2006

You may use a calculator

1. The Martian day, called a sol, is a little longer than a day on earth at about 24 hours and 40 minutes. There are about 669 sols in one Martian year. If Martians also had a seven day week like ours and little Marty was born on Mars on a Tuesday, what day of the week would it be when Marty celebrates his first birthday? We should assume that Martians celebrate birthdays once per Martian year.

2. Fiona needed to multiply a number that is two more than a multiple of 7 by some other number, n, to get a number that is three more than a multiple of 7. In other words, she was trying to solve the equation $2n = 3$ in Mod 7. What whole number value between 0 and 7 will work for n?

3. Find the sum of the first 20 terms of the arithmetic sequence below.

35, 47, 59, 71, ...

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #4, February 2006

Answers

1. Saturday

2. 5

3. 2980

1. Since there are 669 Martian days in a year, there will be 669 ÷ 7 = 95.571428... weeks per year. The decimal part of this number is better understood as 4 extra days beyond the 95 weeks. This means that Marty's birthday will be four days later in the week. Four days after Tuesday is **Saturday**.

2. We need to solve the equation $2n = 3$ in Mod 7. If we just count by 2's in Mod 7, we get 2, 4, 6, 1, 3, 5, 0, and then it repeats. It is the fifth multiple of 2 that equals 3 in Mod 7, so the value of n we need is **5**. To check this, we calculate 2×5 which is normally 10. Subtracting 7 from 10, we confirm that $2 \times 5 = 3$ in Mod 7.

3. The constant difference between the terms in this arithmetic sequence is 12. By the 20th term, a total of nineteen 12's will have been added to the 35, which is the first term. Thus the 20th term is $35 + 19 \times 12 = 35 + 228 = 263$. Now, to find the sum of the terms, we can add the first and last terms, the second and second-to-last terms, etc., to get the same sum ten times. That sum is 35 + 263 = 298, and $10 \times 298 =$ **2980**.

You may use a calculator

Category 4
Arithmetic
Meet #4, February 2006

1. A new pair of snow skis normally sells for $730 but is on sale for 30% off. How much will a customer have to pay for the skis? Give your answer to the nearest whole number of dollars.

2. The bill for dinner at the restaurant was $37.50. If the customer left a tip of $7.50, what was the percent of the tip? Give your answer to the nearest whole-number percent.

3. Ivan deposited $2000 in a bank account that pays 4.8% annually and compounds monthly. (This means that the interest earned is deposited into the account at the end of every month.) How much money can Ivan expect to earn in interest if the certificate of deposit is for 12 months? In calculating this amount, you should not do any intermediate rounding. You should round only your final answer to the nearest whole number of dollars.

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #4, February 2006

Answers

1. \$511

2. 20%

3. \$98
(NOT \$96)

1. Since the skis are selling for 30% off, the customer will have to pay 100 – 30 = 70% of the normal price. We can compute this amount by multiplying the price by the decimal equivalent of 70% as follows:

$$\$730 \times .7 = \mathbf{\$511}$$

2. To find the percent value of the tip, we divide the tip by the cost of the meal and then multiply by 100% as follows:

$$\frac{\$7.50}{\$37.50} \times 100\% = \mathbf{20\%}$$

Month	Balance
0	2000
1	2008
2	2016.032
3	2024.096128
4	2032.192513
5	2040.321283
6	2048.482568
7	2056.676498
8	2064.903204
9	2073.162817
10	2081.455468
11	2089.78129
12	2098.140415

3. An annual interest rate of 4.8 percent comes to 4.8 ÷ 12 = 0.4 percent monthly. This means that at the end of every month, 4 tenths of one percent of the balance will be credited to the account as interest. When the time comes to calculate the interest at the end of the next month, Ivan will be earning interest not only on his deposit of \$2000, but also on the interest that he has earned. It is this interest on interest that makes compound interest more than simple interest. If Ivan were to keep a monthly log, it might look like the table at left. The balance can be calculated directly using the formula below, where P is the principal (the amount invested), r is the annual interest rate as a decimal, and m is the number of months.

$$P\left(1+\frac{r}{12}\right)^m = 2000 \times \left(1+\frac{0.048}{12}\right)^{12} = 2000 \times 1.004^{12}$$

$$\approx 2000 \times 1.049 = 2098$$

The total interest is the extra **\$98** beyond the \$2000

that Ivan originally deposited.

Category 5
Algebra
Meet #4, February 2006

You may use a calculator

1. Shirley works 40 hours per week and gets paid by the hour. For her first 6 weeks on the job, Shirley was paid at a starting wage. For weeks 7 through 12, she got paid 2 dollars more per hour than the starting wage. By the end of her 12th week, Shirley had earned a total of $4560. How many dollars per hour was Shirley earning at the starting wage? Give your answer in dollars to the nearest hundredth of a dollar.

2. At the Useless Trinkets Gift Shop, 4 blips and 5 clips cost 87 cents, 2 clips and 6 glips cost 52 cents, and 3 blips and 1 glip cost 29 cents. How many cents would 2 blips, 2 clips, and 2 glips cost?

3. Eliot drives to work by a 12-mile route and averages 30 miles per hour. He drives home from work by a 15-mile route and averages 40 miles per hour. What is his average speed for the entire round trip of 27 miles? Give your answer to the nearest tenth of a mile per hour.

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #4, February 2006

Answers

1. \$8.50

2. 48

3. 34.8

1. Shirley worked half the time at the starting wage and half the time at the higher wage. Let's solve for the average wage and then adjust. If she had earned the same amount every week, then she would have earned \$4560 ÷ 12 = \$380 per week. Since she worked 40 hours each week, that's \$380 ÷ 40 = \$9.50 per hour. Her starting wage must have been **\$8.50**, and her increased wage must have been \$10.50.

2. There are many ways to solve a system of equations such as this one. First, we can represent the system algebraically as follow:

$$\begin{cases} 4b + 5c \quad\quad = 87 \\ \quad\quad 2c + 6g = 52 \\ 3b \quad\quad + g = 29 \end{cases}$$

In this case, if we add everything up, we get $7b + 7c + 7g = 168$. This can be rewritten as

$$7(b + c + g) = 168.$$

Dividing both sides by 7, we find that $b + c + g = 24$ cents. Therefore, 2 blips, 2 clips, and 2 glips must cost $2 \times 24 =$ **48** cents.

3. We must figure out the amount of time spent driving at each speed and then the total amount of time. Using the equation *distance equals rate times time* ($D = RT$), we write the equation $12 = 30T$ for the trip to work. Solving for T, we get $T = 12 \div 30 = 0.4$ hours. Similarly, for the return trip, we calculate that the time is $15 \div 40 = 0.375$ hours. The total time is $0.4 + 0.375 = 0.775$ hours. Dividing the total distance of 27 miles by the total time of 0.775 hours, we get an average speed of **34.8** mph to the nearest tenth.

Category 6 *You may use a calculator*
Team Questions
Meet #4, February 2006

1. The fifth and the seventh numbers of an arithmetic sequence are 42 and 68. Find the sum of the first ten numbers in the sequence.

2. What is the sum of all the integer values of x that satisfy $\frac{1}{5} < \frac{x}{17} < \frac{4}{7}$?

3. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find the sum of all the numbers from set A such that the Mod 7 powers of that number yield all the elements in A. For example, the powers of 6 in Mod 7 are $6^1 = 6$, $6^2 = 1$, $6^3 = 6$, $6^4 = 1$, etc. Since the powers of 6 do not yield all the elements in set A, it should *not* be included in the sum. Note: This sum should be the actual sum, not the Mod 7 sum.

4. A bagful of coins weighs 235 grams. If a penny weighs 3 grams, a nickel weighs 5 grams, a dime weighs 2 grams, and a quarter weighs 6 grams, what is the greatest amount of money that could be in the bag? Give your answer in dollars to the nearest hundredth of a dollar.

5. Davey joined Abby, Billy, and Casey outside to play in the snow. He also wore a hat and scarf. How many ways can all four kids exchange hats and scarves so that nobody has their own items and nobody has two items from the same person?

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____
6. _____

6. Using the values the team obtained in questions 1 through 4, evaluate the following expression:

$$\sqrt[3]{A + B + 6C + 100D - 5}$$

Solutions to Category 6
Team Questions
Meet #4, February 2006

Answers

1. 485

2. 39

3. 8

4. $11.61

5. 24

6. 12

1. The difference between the fifth and the seventh terms is 26, so the constant difference between terms must be 13. The first ten terms are -10, 3, 16, 29, 42, 55, 68, 81, 94, and 107. Adding the first and the last terms, the second and the second-to-last, etc., we can make five sums of 97. The sum of these ten terms is thus **485**.

2. Dividing whole numbers by 17 on a calculator, we can quickly determine whether they are greater than 0.2, which is the decimal value of $\frac{1}{5}$, and less than 0.57, which is an approximation of $\frac{4}{7}$. The integer values of x that satisfy these conditions are 4, 5, 6, 7, 8, and 9. Their sum is **39**. Alternatively, we could multiply through by $5 \times 17 \times 7$, which gives us the inequality $119 < 35x < 340$. Dividing by 35, we get $3.4 < x < 9.7$, so x can be whole numbers 4 through 9.

3. The table at right shows the mod 7 powers of the numbers in set *A*. The powers of 3 and 5 yield all the numbers in set *A*. Their sum is **8**.

Base \ Exponent	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1

4. We want to use as many dimes as possible since 3 dimes weigh as much as 1 quarter, but are worth 5 cents more. Dividing $235 \div 2$, we get 117 remainder 1. The greatest amount of money that could be in the bag is 116 dimes with 1 penny for a total of **$11.61**.

5. From Abby's point of view, there are three different choices for the hat she can wear. Once she has chosen a hat, there are two remaining choices for the scarf she can wear. Thus there are $3 \times 2 = 6$ ways Abby can pick a hat and a scarf. That's the easy part. Now we have to consider what the other three kids can do after Abby makes her selection. Suppose Abby chooses Billy's hat and Casey's scarf (ABC). Notice that Davey still has his own hat and his own scarf.
If Davey gives his hat to Billy, then everything else is forced, and this is the result: (1) ABC, BDA, CAD, DCB. If instead Billy gives his hat to Casey, then there are three more ways the hats and scarves can be exchanged. They are
(2) ABC, BAD, CDB, DCA; (3) ABC, BAD, CDA, DCB; and (4) ABC, BCD, CDA, DAB.

Abby's original selection was one of six possible choices. Each of these six choices for Abby gives the other kids four possible ways to exchange hats and scarves. Thus there are $6 \times 4 =$ **24** ways the four kids can exchange hats and scarves so that nobody has their own items and nobody has two items from the same person.

6. Substituting the correct values for *A* through *D*, we get:

$$\begin{aligned}\sqrt[3]{A+B+6C+100D-5} &= \sqrt[3]{485+39+6\cdot 8+100\cdot 11.61-5} \\ &= \sqrt[3]{485+39+48+1161-5} \\ &= \sqrt[3]{1728} \\ &= \mathbf{12}\end{aligned}$$