Meet #3
January 2006

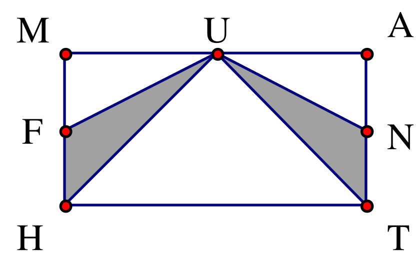
Intermediate Mathematics League of Eastern Massachusetts

Meet #3
January 2006

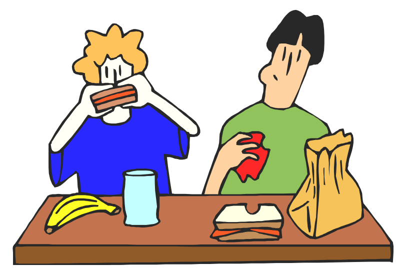
Category 1
Mystery
Meet #3, January 2006

1. Fermat's "two square" theorem assures us that any prime number which leaves a remainder of 1 when divided by 4 can be written as a sum of two squares. For example, the prime number 41 can be written as 16 + 25. Write the prime number 113 as a sum of two perfect square numbers.

2. Points F, U, and N are midpoints on three sides of rectangle MATH. What fraction of the area of this rectangle is occupied by the two shaded regions FUH and NUT? Give your answer in simplest form.



3. Best friends Al, Bob, and Carl have lunch together every day at school. Each boy brings a lunch from home in a lunch box. How many different ways can the three boys swap lunch boxes so that no boy eats his own lunch?



Answers
1. _______ + _______
2. _________________
3. _________________

Solutions to Category 1
Mystery
Meet #3, January 2006

Answers

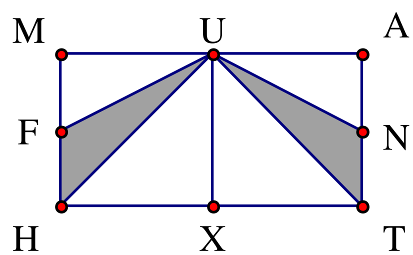
1. 49 + 64 or
64 + 49

2. $\frac{1}{4}$

3. 2

1. We can try subtracting the smallest square numbers from 113 until we get another square that we recognize. It is not until we try 113 – 49 that we get the perfect square 64. Thus 113 = **49 + 64**.

2. It may help to introduce line segment UX, where X is the midpoint of side TH, as shown below. Region MUH occupies half of MUXH, and FUH occupies half of MUH. So region FUH is half of half of half or one eighth of the area of rectangle MATH. Region NUT is another eighth of MATH, so together FUH and NUT occupy $\mathbf{\frac{1}{4}}$ of rectangle MATH.



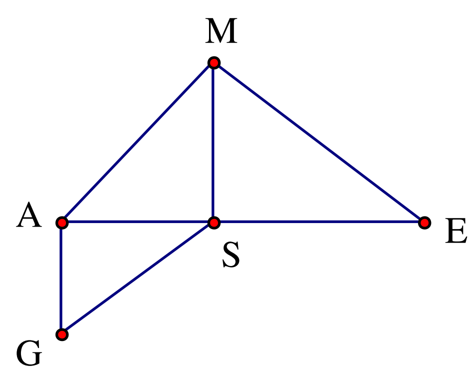
3. Basically we can do **2** rotations. One possibility is Al gets Bob's lunch, Bob gets Carl's lunch, and Carl gets Al's lunch. The other possibility is Al gets Carl's lunch, Bob gets Al's lunch, and Carl gets Bob's lunch.

Category 2
Geometry
Meet #3, January 2006

1. How many degrees are in the measure of an interior angle of a regular 15-gon?

2. A certain polygon has $2\frac{1}{2}$ times as many diagonals as sides. How many sides are there on this polygon? Note: A diagonal is a line segment that connects two non-adjacent vertices of a polygon.

3. Three right triangles are joined together to form the concave pentagon shown below. The measure of GA is 15 units, AS is 20 units, SM is 21 units, and ME is 35 units. How many units are in the perimeter of pentagon GAMES?



Answers
1. ________
2. ________
3. ________

Solutions to Category 2
Geometry
Meet #3, January 2006

Answers

1. 156

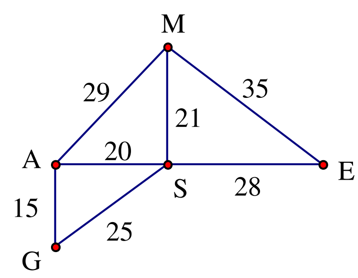
2. 8

3. 132

1. If we draw line segments from one vertex to all the non-adjacent vertices, we can subdivide the 15-gon into 13 triangles. Each triangle has an angle sum of 180 degrees, so the sum of the interior angles of the 15-gon must be $13 \times 180 = 2340$ degrees. Since the 15-gon is regular, this total is shared equally among the 15 interior angles. Each interior angle must have a measure of $2340 \div 15 =$ **156** degrees.

2. The polygon in question must have an even number of sides, since we are multiplying by $2\frac{1}{2}$ and the number of diagonals must be a whole number. A square has only 2 diagonals, so let's try a hexagon. From each vertex of a hexagon we can draw 3 diagonals. If we simply multiply 6×3, we will have counted each diagonal twice, so there must be only $6 \times 3 \div 2 = 9$ diagonals in a hexagon. This is only $1\frac{1}{2}$ times the number of sides. Now let's try an octagon. From each vertex of an octagon we can draw 5 diagonals. That's $8 \times 5 \div 2 = 20$ diagonals in all. Twenty is exactly $2\frac{1}{2}$ times 8, so our polygon has **8** sides.

3. We can use the Pythagorean Theorem to find the lengths of sides AM, GS, and SE. Let the the measure of AM be x. Then $20^2 + 21^2 = x^2$. This means $x^2 = 400 + 441 = 841$. Since $29^2 = 841$, x must be 29 units. Let the measure of GS be y. Then $15^2 + 20^2 = y^2$. (Here we might recognize a multiple of the 3-4-5 Pythagorean Triple.) This means $y^2 = 225 + 400 = 625$. Since $25^2 = 625$, y must be 25 units. Finally, let the measure of SE be z. Then $21^2 + z^2 = 35^2$. This means $z^2 = 35^2 - 21^2 = 1225 - 441 = 784$. Since $28^2 = 784$, z must be 28 units. The perimeter of polygon GAMES is thus 15 + 29 + 35 + 28 + 25 = **132** units.



Category 3
Number Theory
Meet #3, January 2006

1. Express the base three number 121212 as a base nine number.

$$121212_{\text{base three}} = \underline{\qquad\quad}_{\text{base nine}}$$

2. Evaluate the following expression. Write your result in scientific notation.

$$\frac{\left(1.32\times10^{6}\right)\left(1.4\times10^{-7}\right)}{\left(1.1\times10^{-9}\right)}$$

3. Solve the following base-four equation for x. Write your result in base four. Remember that all numbers shown in the equation are base-four numbers.

$$2(13x+22)-12=3200$$

Answers

1. ______________ base nine
2. ____________________
3. ______________ base four

Solutions to Category 3
Number Theory
Meet #3, January 2006

Answers

1. $555_{\text{base niine}}$

2. 1.68×10^8

3. $33_{\text{base four}}$

1. The first six place values in base three are 243, 81, 27, 9, 3, and 1. This means the six-digit number 121212 in base three is equal to $1 \times 243 + 2 \times 81 + 1 \times 27 + 2 \times 9 + 1 \times 3 + 2 \times 1$, which is 455 in base ten. We can convert the expression above directly to base nine, however, without using the base-ten value. The first three place values of base nine are 81, 9, and 1. We can think of the 243 as 3 more 81's, the 27 as 3 more 9's, and the 3 as 3 more 1's. In all, we have five 81's, five 9's, and five 1's, so the base nine number must be $\mathbf{555_{\text{base nine}}}$. If you do convert $455_{\text{base ten}}$ to base nine, you would subtract five 81's, which leaves 50. Then you would subtract five 9's, which leaves 5. Again, you get $\mathbf{555_{\text{base nine}}}$.

2. The expression can be solved as follows:

$$\frac{\left(1.32 \times 10^6\right)\left(1.4 \times 10^{-7}\right)}{\left(1.1 \times 10^{-9}\right)} = \frac{1.32 \times 1.4}{1.1} \times \frac{10^6 \times 10^{-7}}{10^{-9}}$$

$$= \frac{1.848}{1.1} \times 10^{6-7-(-9)}$$

$$= \mathbf{1.68 \times 10^8}$$

3. The equation is solved directly in base four on the left and converted to base ten and back on the right.

$$\begin{aligned} 2(13x + 22) - 12 &= 3200 \\ 32x + 110 - 12 &= 3200 \\ 32x + 32 &= 3200 \\ 32(x+1) &= 3200 \\ x + 1 &= 100 \\ x &= 100 - 1 \\ \mathbf{x} &= \mathbf{33_{\text{base four}}} \end{aligned}$$

$$\begin{aligned} 2(13x + 22) - 12 &= 3200_{\text{base four}} \\ 2(7x + 10) - 6 &= 224_{\text{base ten}} \\ 14x + 20 - 6 &= 224 \\ 14x + 14 &= 224 \\ 14(x+1) &= 224 \\ x + 1 &= 16 \\ x &= 15_{\text{base ten}} \\ \mathbf{x} &= \mathbf{33_{\text{base four}}} \end{aligned}$$

Category 4
Arithmetic
Meet #3, January 2006

1. How many multiples of 3 are there between $\sqrt[3]{26}$ and $\sqrt[3]{2006}$?

2. Evaluate the expression below to the nearest whole number.

$$\sqrt[3]{3^6 - 6^3 - 1^0}$$

3. Evaluate the expression below. Write your answer as a mixed number with the fraction part in lowest terms.

$$\left(2\frac{2}{5}\right)^{-1} + \left(1\frac{1}{2}\right)^{2} + \left(\frac{4}{5}\right)^{-2}$$

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 4
Arithmetic
Meet #3, January 2006

Answers

1. 4

2. 8

3. $4\frac{11}{48}$

1. Since $2^3 = 8$ and $3^3 = 27$, the value of $\sqrt[3]{26}$ must be between 2 and 3. Since $12^3 = 1728$ and $13^3 = 2197$, the value of $\sqrt[3]{2006}$ must be between 12 and 13. Therefore, there are four (**4**) multiples of 3 between $\sqrt[3]{26}$ and $\sqrt[3]{2006}$—3, 6, 9, and 12.

2. The expression can be evaluated as follows:

$$\sqrt[3]{3^6 - 6^3 - 1^0} = \sqrt[3]{729 - 216 - 1} = \sqrt[3]{512} = \sqrt[3]{2^9} = \sqrt[3]{\left(2^3\right)^3} = 2^3 = \mathbf{8}$$

3. We should convert the mixed numbers to simple fractions, since it will be easier to find the reciprocals. The expression can then be evaluated as follows:

$$\begin{aligned}\left(2\frac{2}{5}\right)^{-1} + \left(1\frac{1}{2}\right)^{2} + \left(\frac{4}{5}\right)^{-2} &= \left(\frac{12}{5}\right)^{-1} + \left(\frac{3}{2}\right)^{2} + \left(\frac{4}{5}\right)^{-2} \\ &= \left(\frac{5}{12}\right)^{1} + \left(\frac{3}{2}\right)^{2} + \left(\frac{5}{4}\right)^{2} \\ &= \frac{5}{12} + \frac{9}{4} + \frac{25}{16} \\ &= \frac{20}{48} + \frac{108}{48} + \frac{75}{48} \\ &= \frac{203}{48} \\ &= \mathbf{4\frac{11}{48}}\end{aligned}$$

Category 5
Algebra
Meet #3, January 2006

1. How many integer values of n satisfy the inequality below?

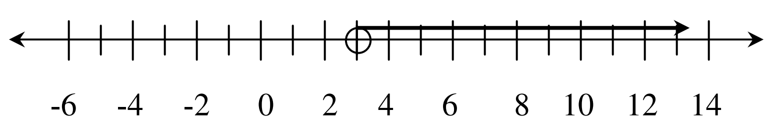
$$\left|\frac{18}{n+1}\right| > 3$$

2. Find the sum of the two solutions to the equation below. Express your answer as a mixed number with the fraction part in lowest terms.

$$|3x-7|+9=42$$

3. For what value of B does the solution to the inequality below match the graph below?

$$3(5x-4)-3x+B>-21$$



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #3, January 2006

Answers

1. 10

2. $4\frac{2}{3}$

3. –45

1. There are ten (**10**) integer values of n that satisfy the inequality. They are -6, -5, -4, -3, -2, (not –1), 0, 1, 2, 3, and 4. For example, when n = -6, we get $\left|\frac{18}{-6+1}\right| = \left|\frac{18}{-5}\right| = \frac{18}{5} = 3\frac{3}{5}$, which is greater than 3.

2. First let's subtract 9 from both sides of the equation.

$$|3x-7| = 33$$

The expression inside the absolute value bars could be either negative or positive, so we have to solve the two separate equations below.

$$\begin{aligned} 3x-7&=33 \\ 3x&=40 \\ x&=\frac{40}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 3x-7&=-33 \\ 3x&=-26 \\ x&=-\frac{26}{3} \end{aligned}$$

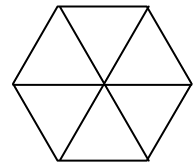
The sum of these two solutions is $\frac{40}{3} + \left(-\frac{26}{3}\right) = \frac{14}{3} = \mathbf{4\frac{2}{3}}$.

3. Let's simplify the inequality first.

$$\begin{aligned} 3(5x-4)-3x+B &> -21 \\ 15x-12-3x+B &> -21 \\ 12x-12+B &> -21 \\ 12x &> -21+12-B \\ 12x &> -9-B \\ x &> \frac{-9-B}{12} \end{aligned}$$

We know from the graph that $x > 3$. This means that $\frac{-9-B}{12}$ must equal 3, or equivalently $-9 - B = 36$. Solving for B, we get $B = -9 - 36 =$ **–45**. Alternatively, one could substitute 3 for x from the beginning and then solve for B.

Category 6
Team Questions
Meet #3, January 2006



1. How many different patterns can be created if anywhere from zero to all six of the triangles in the hexagon at right are shaded black? We will consider rotations to be the same pattern, but mirror images to be different patterns.

2. Dana has $5.00 in quarters and dimes. If he had all quarters, there would be 20 coins. If he had all dimes, there would be 50 coins. How many numbers between 20 and 50 are *not* possible for the total number of coins that Dana has?

3. A Pythagorean Triple is a set of three natural numbers that satisfy the Pythagorean Theorem $a^2 + b^2 = c^2$. A Pythagorean Triple can be generated from a pair of numbers m and n as follows: $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. Given the Pythagorean Triple (39, 80, 89), find the sum of the values of m and n that generated this triple.

4. If the year 2006 is written in all the bases from base two to base nine inclusive, what is the total number of digits written?

5. Best friends Al, Bob, and Carl have decided to let Dudley join their lunch group. Dudley also brings a lunch from home in a lunch box, just like the other three boys. How many different ways can these four boys swap lunch boxes so that no boy eats his own lunch?

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression.

$$\frac{AD}{A + 2B + 3C + 4D + 5E}$$

Solutions to Category 6
Team Questions
Meet #3, January 2006

Answers

1. 14

2. 20

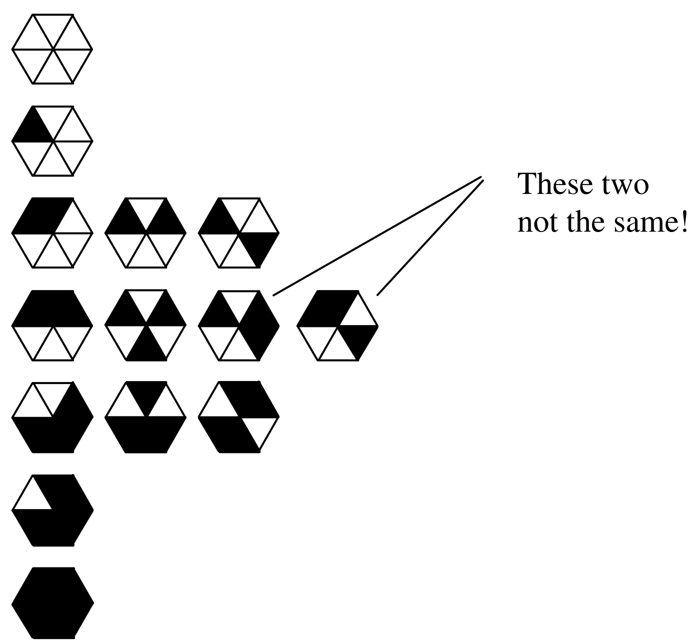
3. 13

4. 46

5. 9

6. 2

1. The fourteen (**14**) possible patterns are shown below. Note the symmetry of the patterns. Note also that there is a left-handed and a right-handed version of the hexagon with two adjacent black triangles and one non-adjacent black triangle.



2. Let's start with 20 quarters and work our way up. We can swap 2 quarters for 5 dimes. That makes 18 quarters and 5 dimes which is 23 coins. Every time we swap 2 quarters for 5 dimes, we get 3 more coins, so the possible numbers of coins are 23, 26, 29, 32, 35, 38, 41, 44, 47, and 50. Twenty (**20**) of the 30 numbers from 21 to 50 are left out.

3. If we add the equations for a and c, we get an equation in one variable $a + c = (m^2 - n^2) + (m^2 + n^2) = 2m^2$. In our example, we have $39 + 89 = 128 = 2m^2$. Dividing by 2 and square rooting, we find that $m = 8$. Now we can solve the equation $b = 2mn$ for n since we know b and m: $80 = 2 \cdot 8 \cdot n$, so $n = 5$. The desired sum $m + n$ is thus $8 + 5 =$ **13**.

4. You do not need to convert 2006 to each base to answer this question. Since $2^{10} = 1024$ and $2^{11} = 2048$, we know that 1024 is the largest place value we will need for the base two version of 2006. Thus, the base-two version of 2006 will be an 11-digit number. (Remember that the ones place is always the base to the zero power.) Similarly, since $3^6 = 729$ and $3^7 = 2187$, the base-three version of 2006 will require 7 digits. The base-four version will require 6 digits. The base-five version and the base-six version will require 5 digits each. The versions for base seven, base eight, and base nine will each require 4 digits. Thus we will write a total of $11 + 7 + 6 + 2 \times 5 + 3 \times 4 = 24 + 10 + 12 =$ **46** digits.

5. Let's say Al picks first. He can choose any of 3 different lunch boxes. If he pick's Bob's lunch box, then Bob will be free to choose any of the other 3 lunch boxes. After Bob chooses, however, Carl and Dudley will have no choice in the matter. If Bob chooses Al's lunch box, then Carl and Dudley must switch so that nobody has their own lunch. If Bob chooses Carl's lunch box, then Carl must take Dudley's so that Dudley can take Al's. Likewise if Bob chooses Dudley's lunch box, then Dudley must take Carl's so that Carl can take Al's. Thus there are 3×3 = **9** possible ways the four boys can share their lunches so that no boy eats his own lunch.

If we consider ABCD to represent each boy getting his own lunch box, then we are looking for all the ways we can scramble these four letters so that no letter is in its starting place. The nine (**9**) ways this can happen are as follows: BADC, BCDA, BDAC, CADB, CDAB, CDBA, DABC, DCAB, and DCBA.

6. Substituting the correct values into the expression and evaluating, we get

$$\begin{aligned}\frac{AD}{A+2B+3C+4D+5E} &= \frac{14\cdot 46}{14+2\cdot 20+3\cdot 13+4\cdot 46+5\cdot 9}\\ &= \frac{644}{14+40+39+184+45}\\ &= \frac{644}{322}\\ &= \mathbf{2}\end{aligned}$$