Meet #4
February 2005

Intermediate Mathematics League of Eastern Massachusetts

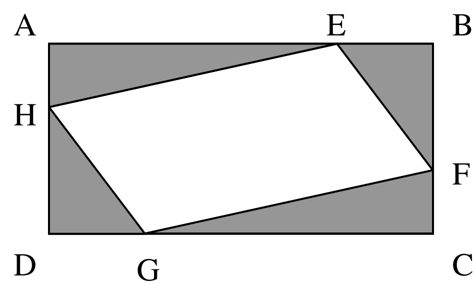
Average team score: 99.1
Average meet for the season: 95.9

Meet #4
February 2005

Category 1
Mystery
Meet #4, February 2005

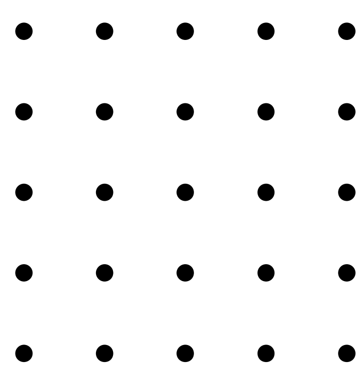
You may use a calculator

1. In the picture below, quadrilateral ABCD is a rectangle and quadrilateral EFGH is a parallelogram. Vertices H and F of the parallelogram meet the short sides of the rectangle one third of the way up or down those sides. Vertices E and G meet the longer sides of the rectangle one quarter of the way from the left or right end of those sides. What fraction of the rectangle is shaded? Express your answer as a common fraction in simplest terms.



2. Since Cynthia's father weighs five times as much as Cynthia, he decides to eat five times as many cookies as Cynthia. If they eat a combined total of 30 cookies, how many more cookies does Cynthia's father eat than Cynthia?

3. How many different lines contain exactly three points of the square 5 by 5 grid shown below?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1 Average team got 13.73 points, or 1.1 questions correct
Mystery
Meet #4, February 2005 Average number of correct answers: 1.14 out of 3

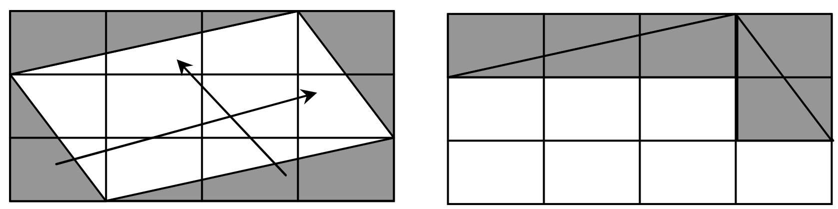
Answers

1. $\frac{5}{12}$

2. 20

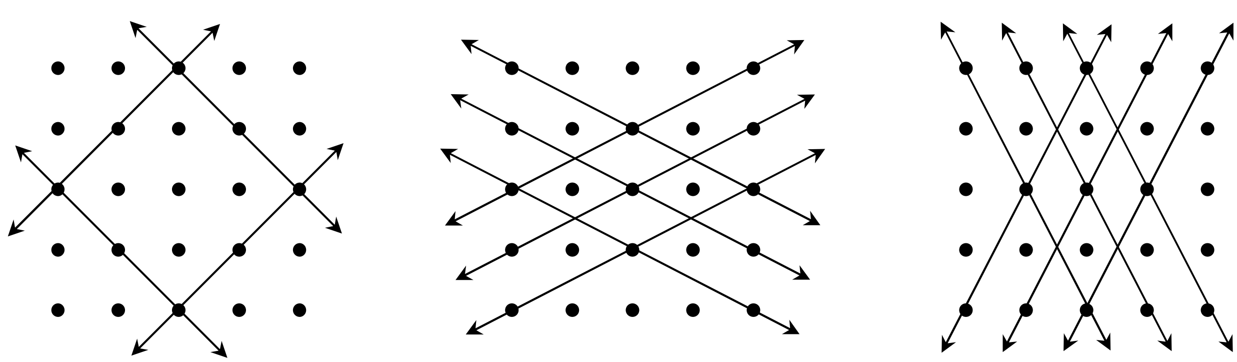
3. 16

1. If we divide the rectangle in thirds horizontally and in forths vertically, we get twelve equal regions. Now we can imagine sliding pairs of congruent triangles together to make two rectangles as shown by the arrows. The areas of these two rectangles are 3 and 2 of the equal regions, for a total shaded area of 5 out of 12 regions. Expressing this as a fraction we get $\mathbf{\frac{5}{12}}$.



2. For every 1 cookie Cynthia eats, her father eats 5 cookies, which is a total of 6 cookies. If this happens five times, then Cynthia will eat 5 cookies and her father will eat 25 cookies for a total of 30 cookies. Cynthia's father will eat 25 – 5 = **20** more cookies than Cynthia.

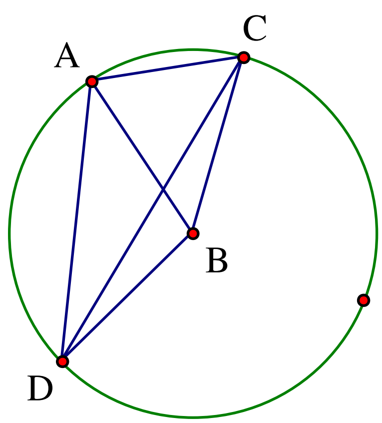
3. The sixteen (**16**) different lines that contain exactly three points of the square 5 by 5 grid are shown below in three sets.



Category 2
Geometry
Meet #4, February 2005

You may use a calculator

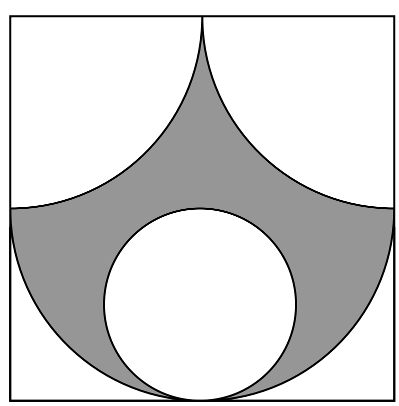
1. A certain circle has a circumference of 125.6 feet. How many square feet are in the area of the circle? Use 3.14 for pi and round your result to the nearest whole number.



2. In the figure at left, point B is the center of the circle. The measure of angle ACB is 65 degrees and the measure of angle BCD is 14 degrees. How many degrees are in the measure of angle ADC?

3. The side length of the square in the figure below is 2 inches. The arcs are all 90 degree arcs of circles with radius 1 inch and the small white circle has a diameter of 1 inch. How many square inches are in the area of the shaded region? Use 3.14 for pi and round your result to the nearest tenth of a square inch.

Answers
1. ______________
2. ______________
3. ______________



Solutions to Category 2 Average team got 12.85 points, or 1.1 questions correct
Geometry
Meet #4, February 2005 Average number of correct answers: 1.07 out of 3

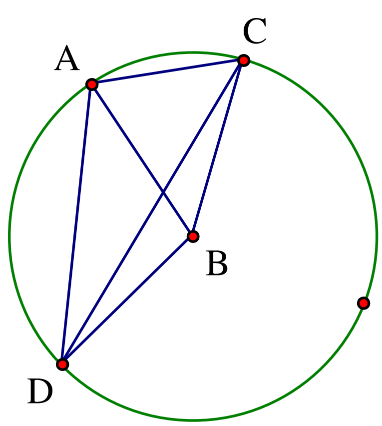
Answers

1. 1256

2. 25

3. 1.2

1. The circumference of a circle is given by the formula $C = \pi D$ where D is the diameter or by $C = 2\pi r$ where r is the radius of the circle. We are given the circumference, so we can divide by pi to find the diameter or twice the radius. $125.6 \div \pi \approx 125.6 \div 3.14 = 40$. So the diameter is 40 feet and the radius is half that, or 20 feet. The area of a circle is given by the formula $A = \pi r^2$. Our circle must have an area of $\pi \times 20^2$ or about 3.14×400 which is **1256** square feet.

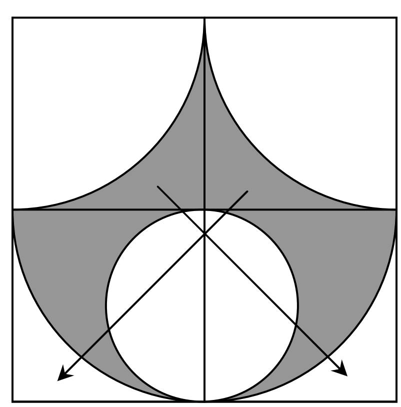


2. Sides AB and BC are radii of the circle, so triangle ABC is isosceles. This means angle BAC has the same measure as angle ACB, namely 65 degrees. Together angles BAC and ACB account for 130 degrees, so the measure of angle ABC must be 180 – 130 = 50 degrees. Angle ADC is known as an iscribed angle and has exactly half the measure of the central angle that intercepts the same arc. Therefore, the measure of angle ADC is $50 \div 2 =$ **25** degrees. Note that we do not need to know the measure of angle BCD.

3. One way to find the area of the shaded region is to divide the square into quarters. We can then see that the two shaded portions in the top half of the square fit perfectly in the two unshaded regions of the lower half of the square, as shown by the arrows. Now we only need to subtract the area of the small circle from half the area of the square. This gives

$\frac{1}{2} A_{\text{square}} - \pi r^2 = \frac{1}{2} \cdot 4 - 3.14 \cdot (0.5)^2 = 2 - 0.785$

= 1.215 or **1.2** to the nearest tenth of a square inch.



Category 3
Number Theory
Meet #4, February 2005

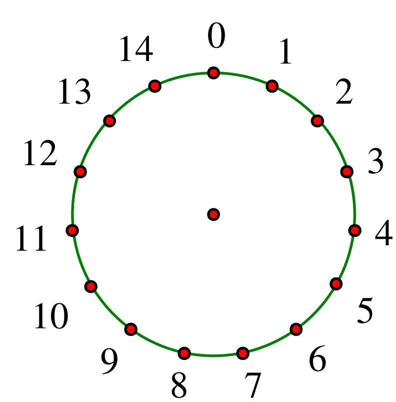
You may use a calculator

1. The measures, in degrees, of the three angles of a triangle form an arithmetic sequence. How many degrees are in the measure of the second largest angle? (Reminder: The sum of the angles of a triangle is 180 degrees.)

2. Find the sum of the first 40 terms in the arithmetic sequence below:

1, 3, 5, 7, 9,...

3. List all the solutions to the equation $6x + 5 = 14$ in Modulo 15, where x is limited to the whole numbers from 0 to 14. (Reminder: The Modulo 15 value of a number is the remainder when the number is divided by 15.)



Answers

1. _______________
2. _______________
3. $x =$ ____________

Solutions to Category 3 Average team got 17.27 points, or 1.4 questions correct
Number Theory
Meet #4, February 2005 Average number of correct answers: 1.44 out of 3

Answers

1. 60

2. 1600

3. 4, 9, 14

1. The second largest angle will be the middle of the three terms in the arithmetic sequence. Let's call it x. Let's call the constant difference between the terms of the sequence d. Then the first term is $x - d$ and the third term is $x + d$. The sum of these three terms must be 180, since they are angles in a triangle. Thus we have $180 = x - d + x + x + d = 3x$. So x must be $180 \div 3 =$ **60** degrees.

2. Some students may already know that the sum of consecutive odds form square numbers. This gives a short-cut to the answer, $40^2 =$ **1600**. Otherwise, we have to use the usual trick of adding first and last numbers in the sequence, multiplying this sum by the number of terms, and dividing by two. The 40th odd number is $2 \times 40 - 1 = 79$, so our sum is:
$(1+79)\times 40 \div 2 = 80 \times 20 = 1600$.

3. Subtracting 5 from both sides of the equation $6x + 5 = 14$ (Mod 15), we get the simpler equation $6x = 9$ (Mod 15). We need to find the multiples of six that are nine more than a multiple of fifteen. If we count by sixes in Mod 15, we get 6, 12, 3, 9, 0, and then the pattern repeats. The fourth multiple of 6 is 9 in Mod 15, because $4 \times 6 = 24$ and $24 - 15 = 9$. Countinuing the pattern, we find that the 9th and the 14th multiples of 6 are also 9. To check this we compute as follows:

$$9 \times 6 = 54 \text{ and } 54 - 3 \times 15 = 9$$
$$14 \times 6 = 84 \text{ and } 84 - 5 \times 15 = 9$$

So the three solutions are $x =$ **4, 9, and 14**.

Category 4
Arithmetic
Meet #4, February 2005

You may use a calculator

1. What is 135% of $\frac{4}{9}$? Express your answer as a common fraction in simplest form.

2. The store "Mathletes 'R' Us" sells sweatshirts with different proofs of the Pythagorean Theorem. The manager purchases the sweatshirts from a distributor at a cost of $12.57 each. She marks up the price so that the price on the tag of each sweatshirt is 60% cost and 40% profit. At the cash register, the customer then has to pay a 5% sales tax on the tag price of the sweatshirt. How much does a customer end up paying for a sweatshirt with a proof of the Pythagorean Theorem? Give your answer in dollars to the nearest hundredth of a dollar.

3. Ron's grandmother sent him a check for $5000 and told him to deposit it in a money market account that earns interest at an annual rate of 3% and compounds monthly. Ron allowed 4 months to go by before he deposited the check. How much interest income did Ron lose by not depositing the check immediately? Express your result to the nearest hundredth of a dollar. Note: This particular money market keeps track of the nearest millionth of a dollar for purposes of calculating interest owed.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4 Average team got 13.96 points, or 1.2 questions correct
Arithmetic
Meet #4, February 2005 Average number of correct answers: 1.16 out of 3

Answers

1. $\frac{3}{5}$

2. \$22.00

3. \$50.19

1. This problem is easier if we convert 135% into the equivalent fraction $\frac{27}{20}$. Now we multiply as follows:

$$\frac{27}{20} \times \frac{4}{9} = \mathbf{\frac{3}{5}}.$$

2. To mark up the sweatshirt so that the tag price is 60% cost and 40% profit, the manager asks herself the following question: \$12.57 is 60% of what number? This corresponds to the equation: $12.57 = 0.6x$. The manager has learned to divide by 0.6 to answer the question. This gives a price of $\$12.57 \div 0.6 = \$20.95$ for the tag price of the shirt. The customer then has to pay a 5% sales tax on this tag price. We can compute the 5% by multiplying \$20.95 by 0.05, which is \$1.0475 or about \$1.05. We then add this tax to the tag price for a total of \$20.95 + \$1.05 = \$22.00. Alternatively, we can find 105% of \$20.95 directly by multiplying by 1.05. We get \$20.95 × 1.05 = \$21.9975, which rounds to nearest cent as **\$22.00**.

3. If the annual interest rate of the money market account is 3%, then the monthly interest rate must be 3% ÷ 12 = 0.25%. After interest is posted at the end of the first month, Ron's balance would be \$5000 × 1.0025 = \$5012.50. Then he begins to earn interest on the interest as well as the original money from his grandmother. After the second month, his balance would be \$5012.50 × 1.0025 = \$5025.03125. After the third month, the balance would be \$5037.593828. Finally, after four months, Ron's balance would be \$5050.187813, which rounds to **\$50.19** to the nearest hundredth of a dollar.

Category 5 *You may use a calculator*
Algebra
Meet #4, February 2005 Average number of correct answers: 1.60 out of 3

1. What negative value of x will make the following proportion true?

$$\frac{121}{x} = \frac{x}{144}$$

2. In parallelogram MATH, MA = 168 mm, AT = $15x + 3$ mm, TH = $3xy^3$ mm, and MH = 108 mm. What is the numerical value of y^x?

3. Together Jim and Bob weigh 357 pounds. Together Jim and Larry weigh 393 pounds. The combined weight of all three men is 565 pounds. How much do Bob and Larry weigh together?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 Average team got 19.15 points, or 1.6 questions correct
Algebra
Meet #4, February 2005

Answers

1. –132

2. 128

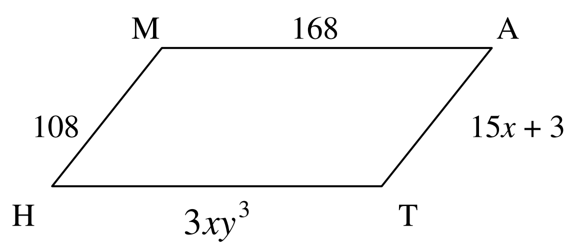
3. 380

1. Cross multiplying, we can turn the proportion into the equation $121 \cdot 144 = x^2$. Each of the numbers on the left is a perfect square, so we can rearrange factors without ever multiplying 121 by 144:

$$121 \cdot 144 = 11 \cdot 11 \cdot 12 \cdot 12 = 11 \cdot 12 \cdot 11 \cdot 12 = 132 \cdot 132 = x^2$$

The value of x could be positive or negative. The question asks for a negative value that will make the proportion true, so the answer is **–132**.

2. Since opposite sides of a parallelogram have the same length, we can write two equations from the information given: $108 = 15x + 3$ and $168 = 3xy^3$. Solving the first equation for x, we get $105 = 15x$, so $x = 105 \div 15 = 7$. Substituting 7 in place of x in the second equation, we get $168 = 21y^3$. This means $y^3 = 168 \div 21 = 8$, so $y = 2$. Finally, the numerical value of y^x is $2^7 =$ **128**.



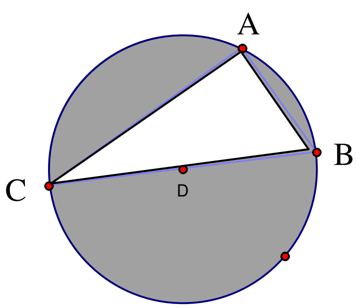
3. There are many ways to solve this system of equations. One clever way is to double the combined weight of all three men and subtract the two paired weights. Since Jim is included in both the paired weights, his weight will be subtracted twice and each of Bob's and Larry's weights will be subtracted once. This leaves the combined weight of Bob and Larry, which is what we want. Thus, the answer is $2 \times 565 - 357 - 393 = 1130 - 357 - 393 =$ **380** pounds. Notice that we did not need to find the individual weights, but, for those interested, Larry weighs 208 pounds, Bob weighs 172 pounds, and Jim weighs 185 pounds.

You may use a calculator

Category 6
Team Questions
Meet #4, February 2005

1. The interior angles of a quadrilateral form an arithmetic sequence. If the measure of the largest angle is 132 degrees, how many degrees are in the measure of the second largest angle?

2. The square root of the positive difference between the cube of a whole number and the square of the same number is 30. What is the number?



3. In the figure at left, segment BC is a diameter of the circle. Segment AB has a length of 8 units and segment AC has a length of 15 units. How many square units are in the area of the shaded region? Round your result to the nearest whole number of a square units.

4. The whole numbers from 0 to 39 are placed in a hat and one number is chosen at random. What is the probability that the number chosen is a solution to the equation $x^2 = 1$ in Modulo 40? Express your answer as a common fraction in simplest form. (Reminder: The Modulo 40 value of a number is the remainder when the number is divided by 40.)

5. Stacey's math teacher drops a quiz score at random and then computes the average of the remaining quizzes. Stacey figured out that this average could be $91\frac{1}{3}$, 92, $93\frac{1}{3}$, or $94\frac{1}{3}$, depending on which quiz score is dropped. What is the sum of Stacey's four quiz scores?

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. __________________

6. Using the values the team obtained in questions 1 through 5, find the prime factorization of the following sum:

$$A + B + C + 40D + E$$

Solutions to Category 6 Average team got 21.92 points, or 3.7 questions correct
Team Questions
Meet #4, February 2005 Average number of correct answers: 3.65 out of 6

Answers

1. 104

2. 10

3. 167

4. $\frac{1}{5}$

5. 371

6. $2^2 \cdot 3 \cdot 5 \cdot 11$
or
$2 \cdot 2 \cdot 3 \cdot 5 \cdot 11$

1. We know that the largest term in the arithmetic sequence is 132. Let's use D to designate the constant difference between terms. Then the second largest term in the sequence is $132 - D$, and the two terms before that are $132 - 2D$ and $132 - 3D$. We also know that all four of these terms must have a sum of 360 degrees, as do all quadrilaterals. Thus we can write the equation:

$$(132-3D)+(132-2D)+(132-D)+132=360$$

We can then solve for D as follows:

$$528-6D=360$$

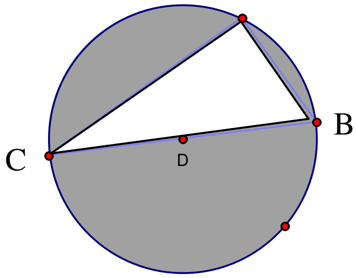
$$528-360=6D$$

$$168=6D$$

$$D=28$$

The second largest angle has a measure of 132 – 28 = **104** degrees.

2. Translating the English to algebra, we get the equation $\sqrt{n^3-n^2}=30$. (Since n is a whole number, the difference under the radical is not negative.) Squaring both sides of the equation, we get $n^3-n^2=900$. At this point we can either start searching by trial and error or we can factor as follows: $n^2(n-1)=900$. We might want to look at the prime factorization of 900, which is $2^2 \cdot 3^2 \cdot 5^2$. Can we rearrange these factors to fit the pattern $n \cdot n \cdot (n-1)$? We can as follows: $(2 \cdot 5) \cdot (2 \cdot 5) \cdot (3 \cdot 3) = 10 \cdot 10 \cdot 9$. The number is thus **10**. To confirm, we calculate as follows: $\sqrt{10^3-10^2}=\sqrt{1000-100}=\sqrt{900}=30$, which matches the original statement exactly.



3. Since segment BC is a diameter, arc BC is 180 degrees, and angle A is half that or 90 degrees. This allows us to use the Pythagorean Theorem to find the diameter of the circle as follows: $D = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$ units. The area of the circle is thus $A_{\text{circle}} = \pi r^2 = 3.14 \times 8.5^2 = 3.14 \times 72.25 = 226.865$ square units. The area of the triangle is $A_{\text{triangle}} = \frac{1}{2}bh = \frac{1}{2} \cdot 8 \cdot 15 = 60$ square units. Subtracting the area of the triangle from that of the circle, we get the area of the shaded region, which is 226.865 – 60 = 166.865 square units. Rounding this to the nearest whole number, we get **167** square units.

4. We need to find out how many of the numbers 0 through 39 are solutions to the equation $x^2 = 1$ in Modulo 40. We could square all 40 numbers and see if these squares have a remainder of 1 when divided by 40. It might be obvious that the solutions must be odd numbers, which would narrow it down to 20 numbers. Another way to find the solutions is to consider numbers that are 1 more than a multiple of 40 and look for perfect squares. These numbers are 1, 41, 81, 121, 161, 201, 241, 281, 321, 361, 401 etc. The perfect squares among these are 1, 81, 121, and 361. Note also that each of these four perfect squares gives us *two* solutions, the positve and the negative value, so there are **8** solutions in all. (–1 = 39 in Modulo 40.) The eight solutions are x = 1, 9, 11, 19, 21, 29, 31, and 39. Thus the probability that a solution is chosen at random is $\frac{8}{40} = \mathbf{\frac{1}{5}}$.

5. If we were to triple each of the averages of three quiz scores, we would have the sum of each set of three. The sum of these four sums would include each of the quizzes three times. But why bother to triple and then divide by three. The sum of the four averages given is exactly the same as the sum of the four quiz scores. Thus the answer is $91\frac{1}{3} + 92 + 93\frac{1}{3} + 94\frac{1}{3} = \mathbf{371}$.

6. Substituting the correct values for A through E, we get the following sum:

$$A + B + C + 40D + E = 104 + 10 + 167 + 40 \cdot \frac{1}{5} + 371 = 660.$$

The prime factorization of 660 is $\mathbf{2^2 \cdot 3 \cdot 5 \cdot 11}$ or $\mathbf{2 \cdot 2 \cdot 3 \cdot 5 \cdot 11}$.