Meet #2
November 2004

Intermediate Mathematics League of Eastern Massachusetts

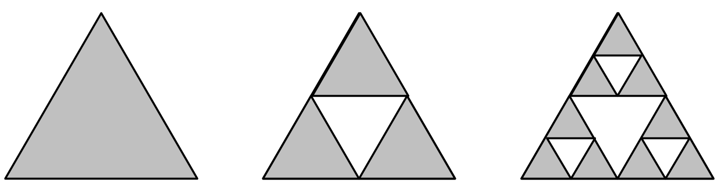
Average team score: 70.57
Average meet for the season: 95.9

Meet #2
November 2004

Category 1
Mystery
Meet #2, November 2004

1. A classroom has three light switches, one for each row of lights on the ceiling. Including the possibility of all three switches off, how many different ways can the lights be set in this classroom?

2. At stage 0, the triangle below has an area equal to 1 square unit. At stage 1, a similar triangle has been removed from the middle, and the area is now equal to 3/4 of a square unit. At each new stage, the middles of the remaining triangles are removed, and the area is 3/4 of what it was in the previous stage. How many square units are in the area of the remaining shaded regions at stage 4? Express your answer as a common fraction in lowest terms.



Stage 0 Stage 1 Stage 2

3. Let's say that a "comprimesite" number is a product of exactly two primes. The number 6, for example, is a comprimesite number, since it can be written as 2×3. The number 4 is also comprimesite, since it can be written as 2×2. The numbers 7 and 8 are not comprimesite, since they cannot be written as a product of two primes. Find the greatest comprimesite number less than 200.

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 1 Average team got 13.09 points, or 1.1 questions correct
Mystery
Meet #2, November 2004

Answers

1. 8

2. $\dfrac{81}{256}$

3. 194

1. Each of the three light switches can be either on or off. Thus there are $2 \times 2 \times 2 =$ **8** ways the lights can set in the classroom. If we use 1 for "on" and 0 for "off", we can represent the 8 different ways as three-digit binary numbers as follows: 000, 001, 010, 011, 100, 101, 110, 111.

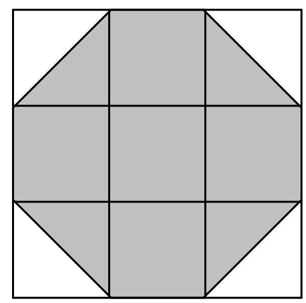
2. At stage 4, the remaining area will be
$\left(\frac{3}{4}\right)^4 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \mathbf{\frac{81}{256}}$.

3. The greatest "comprimesite" number less than 200 is **194**, which is 2×97.

Category 2
Geometry
Meet #2, November 2004

1. How many different rectangles have a perimeter of 48 units, if both the length and the width of the rectangle are positive whole numbers?
Note: A rectangle with length *a* and width *b* is considered the same as a rectangle with length *b* and width *a*.

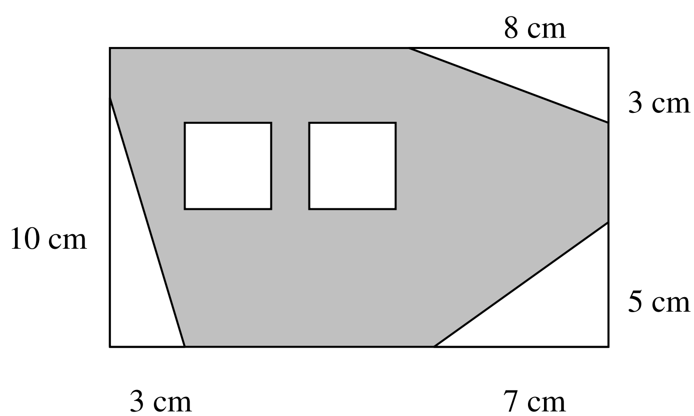
2. An irregular octagon is made on a grid as shown at right. Notice that some sides are longer than others. If the octagon has an area of 252 square inches, how many inches are in the length of a short side?



3. Three right triangles and two squares were cut out of a rectangle with length 20 centimeters and width 12 centimeters as shown below. The dimensions of the triangles are given in the picture and the squares are each 3.5 cm by 3.5 cm. How many square centimeters are in the area of the shaded region?

Answers

1. ________________
2. ________________
3. ________________



Solutions to Category 2 Average team got 13.74 points, or 1.1 questions correct
Geometry
Meet #2, November 2004

Answers

1. 12

2. 6

3. 171

1. Since the perimeter of a rectangle is twice the sum of the length and the width, we will consider all whole number lengths and widths with a sum of 24 units.
The **12** possible dimensions of the rectangle are: 1×23, 2×22, 3×21, 4×20, 5×19, 6×18, 7×17, 8×16, 9×15, 10×14, 11×13, and 12×12.

2. There are 5 full squares and 4 half squares in the shaded area, for a total of 7 squares the size of the grid. Since the area of the shaded region is 252 square inches, each square in the grid must be $252 \div 7 = 36$ square inches. This means each grid square must be 6 inches on a side ($6 \times 6 = 36$), and there are also **6** inches in the short side of the octagon.

3. Before the triangles and squares were cut out of the rectangle, it had an area of 12 cm $\times$ 20 cm = 240 square centimeters. The areas of the triangles are $10 \times 3 \div 2 = 15$ cm^2, $8 \times 3 \div 2 = 12$ cm^2, and $5 \times 7 \div 2 = 17.5$ cm^2. The area of the two squares is $2 \times 3.5 \times 3.5 = 24.5$ cm^2. Subtracting these areas from the original 240 square centimeters, we get $240 - 15 - 12 - 17.5 - 24.5 =$ **171** square centimeters.

Category 3
Number Theory
Meet #2, November 2004

1. Find the greatest common factor (GCF) of 5,409,149,074,560 and 37,644,750, given their prime factorizations below.

$$5{,}409{,}149{,}074{,}560 = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^3 \cdot 11^5 \cdot 13^0 \cdot 17^1$$

$$37{,}644{,}750 = 2^1 \cdot 3^4 \cdot 5^3 \cdot 7^0 \cdot 11^1 \cdot 13^2 \cdot 17^0$$

2. If LCM(a,b) = 90 and LCM(a,c) = 36, then what is the value of LCM(a,b,c)?
Note: "LCM(a,b)" means the least common multiple of a and b.

3. If m is a proper factor of 208 and n is a proper factor of 288 and GCF(m,n) = 2, then what is the greatest possible value of $m + n$?
Note: A proper factor is a factor of a number other than the number itself.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3 Average team got 10.34 points, or 0.9 questions correct
Number Theory
Meet #2, November 2004

Answers

1. 990

2. 180

3. 170

The greatest common factor of the two numbers is the product of the primes that they have in common. For each prime factor, we use the lower exponent to create the GCF. Thus the GCF of the two numbers is $2^1 \cdot 3^2 \cdot 5^1 \cdot 7^0 \cdot 11^1 \cdot 13^0 \cdot 17^0 = 2 \cdot 9 \cdot 5 \cdot 11 = 10 \cdot 99 = \mathbf{990}$.

2. The LCM(a,b,c) is the least common multiple of the LCM(a,b) and the LCM(a,c). In other words, we need to find LCM(90,36). It is helpful to consider the prime factors of each number: $90 = 2 \cdot 3^2 \cdot 5$ and $36 = 2^2 \cdot 3^2$. The LCM of these two numbers must have all the prime factors of each and no extras. If we start with the prime factorization of 90, we only need one more factor of 2 to create a multiple of 36. Thus the LCM of 90 and 36—and also of a, b, and c—is 90×2 = **180**.

3. The proper factors of 208 are 1, 2, 4, 8, 13, 16, 26, 52, and 104. The proper factors of 288 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, and 144.
Taking one number from each list and making sure their GCF is 2, we get the greatest possible sum from 144 and 26, namely **170**.

Category 4
Arithmetic
Meet #2, November 2004

1. What is 35% of 0.4 of $\frac{9}{14}$ of $\frac{8}{9}$ of 625?

2. Find the fraction in lowest terms that is equivalent to the repeating decimal $0.2\overline{7}$.

3. What is the 39th digit in the repeating decimal equivalent of $\frac{5}{14}$?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4 Average team got 12.91 points, or 1.1 questions correct
Arithmetic
Meet #2, November 2004

Answers

1. 50

2. $\frac{5}{18}$

3. 7

$$
\begin{array}{r}
0.3\overline{571428} \\
14\overline{)5.0000000} \\
\underline{-42} \\
80 \\
\underline{-70} \\
100 \\
\underline{-98} \\
20 \\
\underline{-14} \\
60 \\
\underline{-56} \\
40 \\
\underline{-28} \\
120 \\
\underline{-112} \\
80
\end{array}
$$

1. It is helpful to rewrite all quantities in fraction form. The percent can be written as $\frac{35}{100}$, which reduces to $\frac{7}{20}$. The decimal 0.4 can be written as $\frac{4}{10}$, which reduces to $\frac{2}{5}$. Since the word "of" means multiply, we can rewrite the expression as $\frac{7}{20} \cdot \frac{2}{5} \cdot \frac{9}{14} \cdot \frac{8}{9} \cdot \frac{625}{1}$. Now we can reduce by cancelling common factors, until we get 2×25 = **50**.

2. We use a clever bit of algebra to convert a repeating decimal to a fraction. Let's say $x = 0.2\overline{7}$. Then $10x = 2.\overline{7}$ and $100x = 27.\overline{7}$. Now we subtract one equation from another as shown below:

$$
\begin{array}{rl}
100x = & 27.\overline{7} \\
\underline{-\ 10x = -} & \underline{2.\overline{7}} \\
90x = & 25
\end{array}
$$

Dividing both sides of the equation by 90 and simplifying, we get:

$$x = \frac{25}{90} = \mathbf{\frac{5}{18}}$$

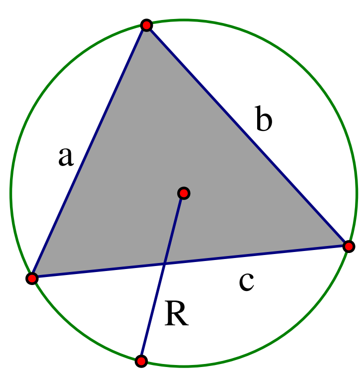
3. After the digit 3 in the tenths place, the decimal expansion of 5/14 has a six-digit repeating cycle, as shown at left. This means that the 39^{th} digit in the repeating decimal will actually be the 38^{th} digit in the repeating part. Since $38 = 6 \times 6 + 2$, it will be the 2^{nd} digit in the cycle, which is a **7**.

Category 5
Algebra
Meet #2, November 2004

1. Five consecutive multiples of 11 have a sum of 1155. What is the product of the second and the fourth of these five numbers?

2. At a school play, tickets for students cost \$3 and tickets for adults cost \$5. If 620 tickets were sold and a total of \$2660 was made, how many adults bought tickets for the play?

3. The formula $R = \dfrac{abc}{4 \cdot A_{\text{Triangle}}}$ can be used to calculate the radius of a circle that is circumscribed around a triangle, where a, b, and c are the side lengths of the triangle and A_{Triangle} is the area of the triangle. The formula $A_{\text{Triangle}} = \sqrt{s(s-a)(s-b)(s-c)}$ can be used to calculate the area of a triangle, where a, b, and c are again the side lengths of the triangle and s is the semiperimeter (half the perimeter). Find the radius of the circle that circumscribes a triangle with side lengths 13, 14, and 15 units. Express your result in lowest terms as a common fraction, not a mixed number.



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 Average team got 9.17 points, or 0.8 questions correct
Algebra
Meet #2, November 2004

Answers

1. 53240

2. 400

3. $\frac{65}{8}$

1. If five consecutive multiples of 11 have a sum of 1155, then their average is 1155 ÷ 5 = 231, which is also the middle number. The five multiples of 11 are: 209, 220, 231, 242, and 253. The product of the second and forth of these is 220 × 242 = **53240**.

2. If the same number of tickets had been sold to students and adults, the average ticket price would have been \$4 and the total ticket sales would have been \$4 × 620 = \$2480. Since the total ticket sales was \$2660, we know that there were more adults than students. The difference \$2660 – \$2480 = \$180, tells us that there were 180 more adult tickets than student tickets. Subtracting these excess adults tickets from the 620 tickets, we get 440 tickets that were sold equally to adults and students. Thus there were 220 student tickets sold and 220 + 180 = **400** adult tickets.

3. First we need to plug the values 13, 14, and 15 into the equation for the area of the triangle. The semiperimeter is s = (13 + 14 + 15) ÷ 2 = 42 ÷ 2 = 21.

$$\begin{aligned} A_{\text{Triangle}} &= \sqrt{21(21-13)(21-14)(21-15)} \\ &= \sqrt{21 \cdot 8 \cdot 7 \cdot 6} \\ &= \sqrt{2^4 \cdot 3^2 \cdot 7^2} \\ &= 2^2 \cdot 3 \cdot 7 \\ &= 84 \end{aligned}$$

Now we can use the first formula to find the radius of the circumscribed circle.

$$R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{13 \cdot 5}{4 \cdot 2} = \mathbf{\frac{65}{8}}.$$

Category 6
Team Questions
Meet #2, November 2004

1. Find the fraction in lowest terms that is equal to the repeating decimal $0.3\overline{28}$.

2. A rectangle has whole number length and width and an area of 48 square units. How many units are in the median of all the possible perimeters of this rectangle?

3. Gina and Tina each have a positive whole number of lollypops. Three eighths of Gina's lollypops is four more than two fifths of Tina's lollypops. If they have the least possible values for which the previous statements are true, how many lollypops do they have together?

4. Recall that a "comprimesite" number is the product of two primes. What is the mean (average) of all the comprimesite numbers between 180 and 200? Express your answer as a decimal to the nearest hundredth.

5. Let's say that a "comprimesite factorization" is a way of expressing numbers as products of "comprimesite" numbers. We will allow at most one prime number in a comprimesite factorization, but all other factors must be comprimesite. It turns out that comprimesite factorizations are not unique. For example, the number 80 can be written as $2 \times 4 \times 10$ or $4 \times 4 \times 5$. How many different comprimesite factorizations are there for 1800? Note: The order in which the factors are written does not matter.

Answers

1. ______________ = A/B
2. ______________ = C
3. ______________ = D
4. ______________ = E
5. ______________ = F
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below. Notice that the numerator and the denominator of the answer to question 1 are taken separately.

$$\sqrt[3]{(2C-A)\left(\frac{E-1/4}{D-4}\right)\left(\frac{2B}{C-2}\right)}+F$$

Solutions to Category 6 Average team got 11.55 points, or 1.9 questions correct
Team Questions
Meet #2, November 2004

Answers

1. $\frac{65}{198}$

2. 38

3. 21

4. 187.25

5. 11

6. 22

1. Let $x = 0.3\overline{28}$. Then $10x = 3.\overline{28}$ and $1000x = 328.\overline{28}$. Now we subtract one equation from another as shown below:

$$\begin{aligned} 1000x &= 328.\overline{28} \\ -\ 10x &= -\ 3.\overline{28} \\ \hline 990x &= 325 \end{aligned}$$

Dividing both sides of the equation by 990 and simplifying, we get:

$$x = \frac{325}{990} = \mathbf{\frac{65}{198}}$$

2. The possible whole number dimensions for a rectangle with 48 square units of area are shown below, along with the corresponding perimeters.

Dimensions	Perimeter
1×48	$49 \times 2 = 98$ units
2×24	$26 \times 2 = 52$ units
3×16	$19 \times 2 = 38$ units
4×12	$16 \times 2 = 32$ units
6×8	$14 \times 2 = 28$ units

The median of these perimeters is **38** units.

3. Converting the English to algebra, we can write the equation $\frac{3}{8}G = 4 + \frac{2}{5}T$, for which we need to find the least possible positive integer solution. There are some advanced techniques for solving this type of equation, but trial and error is likely to be the fastest with these small numbers. Let's try $G = 8$ and $T = 5$. Then we get 3 = 4 + 2, which is false. With $G = 16$ and $T = 5$, we get 6 = 4 + 2, which is true. Together Gina and Tina have 16 + 5 = **21** lollypops.

4. The four "comprimesite" numbers between 180 and 200 are $3 \times 61 = 183$, $5 \times 37 = 185$, $11 \times 17 = 187$, and $2 \times 97 = 194$. The mean of these four numbers is $(183 + 185 + 187 + 194) \div 4 = 749 \div 4 =$ **187.25**.

5. The usual prime factorization of 1800 is $2^3 \cdot 3^2 \cdot 5^2$. The "comprimesite factorizations" of 1800 will all have a single prime factor and three comprimesite factors. The list below shows the **11** possible "comprimesite factorizations" of 1800.

$2 \cdot 4 \cdot 9 \cdot 25$		
$2 \cdot 4 \cdot 15 \cdot 15$	$3 \cdot 4 \cdot 6 \cdot 25$	$5 \cdot 4 \cdot 6 \cdot 15$
$2 \cdot 6 \cdot 6 \cdot 25$	$3 \cdot 4 \cdot 10 \cdot 15$	$5 \cdot 4 \cdot 9 \cdot 10$
$2 \cdot 6 \cdot 10 \cdot 15$	$3 \cdot 6 \cdot 10 \cdot 10$	$5 \cdot 6 \cdot 6 \cdot 10$
$2 \cdot 9 \cdot 10 \cdot 10$		

6. Substituting the correct values for *A* through *F* into the expression, we get:

$$\begin{aligned}
\sqrt[3]{(2C-A)\left(\frac{E-1/4}{D-4}\right)\left(\frac{2B}{C-2}\right)}+F &= \sqrt[3]{(2\cdot 38-65)\left(\frac{187.25-1/4}{21-4}\right)\left(\frac{2\cdot 198}{38-2}\right)}+11 \\
&= \sqrt[3]{(76-65)\left(\frac{187}{17}\right)\left(\frac{396}{36}\right)}+11 \\
&= \sqrt[3]{(11)(11)(11)}+11 \\
&= 11+11 \\
&= \mathbf{22}
\end{aligned}$$