Meet #4
February 2004

Intermediate Mathematics League of Eastern Massachusetts

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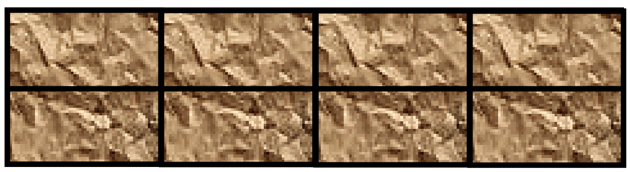
You may use a calculator

Category 1
Mystery
Meet #4, February 2004

1. Bobby had 12 candy bars like the one shown below. Each can be easily broken along the lines into eighths of a bar. He divided the 12 bars into 7 equal groups, so that he could eat the same amount each day for one week. He did not want to make any pieces smaller than one eighth of a bar, so he gave the extra to his younger sister. How much of a candy bar did his sister get? Express your answer as a simple fraction in lowest terms.



2. Julia has 15 books on her reading list. She plans to bring 2 of them with her on a the plane. How many different ways can she bring 2 of these 15 books?

3. Takiya paints $\frac{1}{3}$ of a house in 4 days. Her friend Barbara, who works more slowly, joins in to help, and they finish the job in 5 more days. How long would it have taken Barbara to do the entire job by herself?

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 1
Mystery
Meet #4, February 2004

Answers

1. $\frac{5}{8}$

2. 105

3. 20

1. There are a number of ways to think about this. Let's imagine making seven equal piles. We can definitely put one candy bar in each pile, which leaves 12 – 7 = 5 candy bars. These five candy bars can be broken into 5 × 8 = 40 eighths. Since 7 × 5 = 35, we can put 5 more eighths in every pile. Bobby sister would get the 40 – 35 = 5 eighths of a candy bar left over. Notice that the division problem $12 \div 7 = \frac{12}{7} = 1\frac{5}{7}$ does not match our problem. Some people might convert the twelve candy bars into eighths, which is $12 \times \frac{8}{8} = \frac{96}{8}$. Now, if we take these 96 eighths and write the division with a quotient and a remainder, we will get 96 ÷ 7 = 13 R5. It is these remaining five eighths $\left(\frac{\mathbf{5}}{\mathbf{8}}\right)$ that Bobby's sister gets.

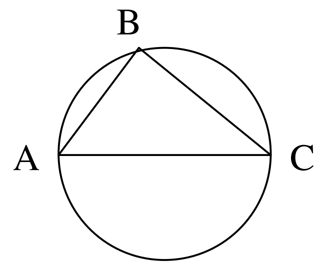
2. This is a question about combinations. Julia could choose any of the 15 books as the first book she brings with her on the plane. For each of these 15 choices, she has 14 choices for the second book, so we multiply 15 × 14 = 210. Now we have to realize that if she chose book *A* followed by book *B*, the result is the same as if she chose book *B* followed by book *A*, i.e. she would have books *A* and *B* with her on the plane and the order in which they were picked would not matter. We must divide our 210 by the number of ways that two things can be chosen, which is 2, so we get 210 ÷ 2 = **105** ways. This can be calculated as follows:

$$ {}_{15}C_2 = \frac{15!}{2!\cdot(15-2)!} = \frac{15\cdot 14\cdot \cancel{13}\cdot \cancel{12}\cdot \cancel{11}\cdot \cancel{10}\cdot \cancel{9}\cdot \cancel{8}\cdot \cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{2\cdot(\cancel{13}\cdot \cancel{12}\cdot \cancel{11}\cdot \cancel{10}\cdot \cancel{9}\cdot \cancel{8}\cdot \cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1})} = \frac{15\cdot 14}{2} = \frac{210}{2} = 105 $$

3. Since Takiya painted $\frac{1}{3}$ of a house in 4 days, it would have taken her 3 × 4 = 12 days to do the job herself. She was working at a rate of $\frac{1}{12}$ of the house per day. With Barbara's help, they finished the other $\frac{2}{3} = \frac{8}{12}$ of the job in 5 days. Takiya can be credited with another $\frac{5}{12}$ of the job, so Barbara must have done $\frac{3}{12} = \frac{1}{4}$ of the job. Since Barbara did $\frac{1}{4}$ of the job in 5 days, it would have taken her 4 × 5 = **20** days to do the entire job by herself.

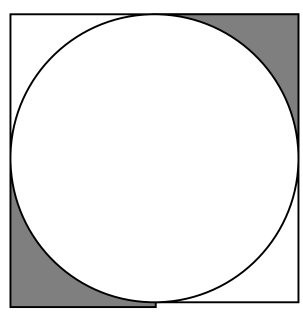
You may use a calculator

Category 2
Geometry
Meet #4, February 2004



1. Segment AC is a diameter of the circle at right. If the measure of angle CAB is 49 degrees, how many degrees are in the measure of angle ACB?

2. A circle is inscribed in a square whose area is 2.25 square inches. How many square inches are there in the sum of the areas of the two shaded regions? Use 3.14 for π and express your answer as a decimal to the nearest hundredth.



3. Find the number of feet in the radius of a circle whose area given in square yards is numerically equivalent to its circumference given in feet.
(Reminder: Three feet equals one yard.)

Answers
1. ________________
2. ________________
3. ________________

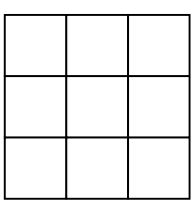
Solutions to Category 2
Geometry
Meet #4, February 2004

Answers

1. 41

2. 0.24 or .24

3. 18



One square yard
equals
nine square feet.

1. Since segment AC is a diameter, angle B must be a right angle. This means that angles A and C must add up to the other 90 degrees in the triangle. 90 – 49 = 41, so the measure of angle ACB must be **41** degrees.

2. If the area of the square is 2.25 square inches, its side length is the square root of 2.25, which is 1.5 inches.

$$\sqrt{2.25} = 1.5 \Leftrightarrow 1.5^2 = 2.25$$

The side length of the square and the diameter of the circle are the same. The formula for the area of a circle is $A = \pi r^2$, so we need the radius of the circle, which is 1.5 ÷ 2 = 0.75 inches. The area of the circle is thus $A_{\text{circle}} = 3.14 \times 0.75^2 = 3.14 \times 0.5625 = 1.76625$ square inches. Subtracting this from the area of the square, we get 2.25 – 1.76625 = 0.48375 square inches. The two shaded regions account for half of this difference, or 0.48375 ÷ 2 = 0.241875 square inches. Rounding this to the nearest hundredth, we get **0.24**.

3. We have *A* = *C*, but *A* is in square yards and *C* is in square feet. Since we would like to write a single equation to solve for the unknown radius, we should convert the area of the circle from square yards to square feet. There are three feet in a yard, but *nine* square feet in one square yard (see picture at left). Our conversion will make the numerical value of the area nine times greater. We will have to multiply the value of the *circumference* by nine to keep these two quantities numerically equal. Thus we have the equation $A = 9C$. Since $A = \pi r^2$ and $C = 2\pi r$, we have $\pi r^2 = 9(2\pi r)$. Dividing both sides by πr, we get r = **18** feet.

You may use a calculator

Category 3
Number Theory
Meet #4, February 2004

1. What is the value of the 75th term in the arithmetic sequence shown below? Note: –63 is the first term of the sequence.

–63, –57, –51, –45,...

2. Find the sum of the first 112 terms in the arithmetic sequence shown below.

93, 96, 99, 102,...

3. Coach McMod gave his fifteen basketball players jerseys with the numbers 0 through 14 on them. For passing practice, the players always form a circle so that their jersey numbers go in clockwise ascending order. During practice last Tuesday, Coach McMod gave the ball to the player with jersey number 6 and told the players to pass to the fourth person on their left. After 71 passes the coach blew the whistle and the passing stopped. What was the number on the jersey of the player holding the ball?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #4, February 2004

Answers

1. 381

2. 29,064

3. 5

1. The terms are increasing by six. There will be 74 increases of 6 between the first and the 75th terms. $6 \times 74 = 444$, so the 75th term will be 444 more than the first term, which is $-63 + 444 =$ **381**. The general formula for the *N*th term of an arithmetic sequence is $Z = A + (N-1)D$, where *Z* is the value of the *N*th term, *A* is the first term, and *D* is the common difference between any two consecutive terms.

2. The constant difference between terms is 3. The 112th term will be $3 \times 111 = 333$ greater than the first term, which is $93 + 333 = 426$. The sum of the first and last terms in this sequence is $93 + 426 = 519$. The sum of the second and the second-to-last terms is also $96 + 423 = 519$. The sum of the third and third-to-last terms is likewise 519. In all there will be $112 \div 2 = 56$ of these sums of 519. This gives a total of $56 \times 519 =$ **29,064**. The general formula for the *sum* of *N* terms of an arithmetic sequence is $S = \frac{N}{2}\left[2A + (N-1)D\right]$, where *S* is the sum of the *N* terms, *A* is the first term, and *D* is the common difference between any two consecutive terms.

3. $71 \times 4 = 284$, $284 \div 15 = 18$ remainder 14. This means that the ball goes 18 times around the circle and ends up 14 people to the left of where it started. Since there are 15 people in the circle, the person 14 people to the left of any player is simply the person on the right. It started with the person wearing jersey number 6, so it ends up with the person wearing jersey number **5**.

You may use a calculator

Category 4
Arithmetic
Meet #4, February 2004

1. What fraction is 125% larger than $\frac{2}{9}$? Express your result as a fraction in lowest terms.

2. Mthomba is a visiting dignitary who is entirely unaware of the American custom of tipping waiters and waitresses a minimum of 15% of the cost of the meal. In his country, the standard practice is to leave the equivalent of 8 U.S. dollars, regardless of the cost of the meal. If Mthomba leaves an $8 tip for a meal that cost $78.60, how much less than the expected minimum tip did he leave? Express your answer in dollars to the nearest cent.

3. Fred invested $5000 in an special account that pays 10% annual interest, compounded annually. What will his account balance be after ten years? Express your answer in dollars to the nearest cent.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4
Arithmetic
Meet #4, February 2004

Answers

1. $\frac{1}{2}$

2. \$3.79

3. \$12,968.71 **

1. If B is 125% *larger than* A, it is equal to 100% of A *plus* 125% of A. It is easier to think of the new number as 100% + 125% = 225% of A. Converting the percent to a decimal and then to a fraction, we get 225% = 2.25 = $\frac{9}{4}$. Now we just need to find $\frac{9}{4}$ of $\frac{2}{9}$, which is:

$$\frac{9}{4}\times\frac{2}{9}=\frac{18}{36}=\mathbf{\frac{1}{2}}.$$

2. To calculate 15% of \$78.60, we convert the percent to a decimal and then multiply by \$78.60. The expected minimum tip is thus $0.15\times\$78.60=\$11.79$. Mthomba left only \$8.00, which is \$11.79 – \$8.00 = \$3.79 less than the expected minimum tip.

Year	Balance
0	\$ 5,000.00
1	\$ 5,500.00
2	\$ 6,050.00
3	\$ 6,655.00
4	\$ 7,320.50
5	\$ 8,052.55
6	\$ 8,857.81
7	\$ 9,743.59
8	\$ 10,717.94
9	\$ 11,789.74
10	**\$ 12,968.71**

3. When 10% interest is payed on an account balance, this amount is added to the balance, so the new balance is 110% of the previous balance. We can simply multiply the previous balance by 1.10 to get the new balance each year. The table at left shows Fred's account balance over ten years. The short-cut way is to simply calculate $\$5000\times1.1^{10}=\mathbf{\$12968.71}$.

The general formula is $B=P(1+r)^n$, where B is the balance, P is the principle (the amount originally invested), r is the annual interest rate, and n is the number of years the interest is compounded. It gets a little more complicated if the interest is compounded more often than once a year.

** Editor note: Other possible answers range from 12,968.69 to 12,968.73 depending upon when and how rounding or truncation occurs. Real banks won't keep your balance with indefinite precision each year, so "round only at the end" is unrealistic.

You may use a calculator

Category 5
Algebra
Meet #4, February 2004

1. Julia and Michelle are best friends who live exactly four miles apart. One day, Julia called Michelle on the phone and said, "I've simply got to tell you the rumor I heard as quickly as possible, and I can't tell you over the phone." Michelle said, "I'll come right over on my bike." Julia said, "I'll start running toward you." If Julia ran toward Michelle at a speed of 6 miles per hour and Michelle rode her bicycle toward Julia at a speed of 14 miles per hour, how many minutes later did they meet? Assume that they set out immediately, and express your answer to the nearest whole number of minutes.

2. A vendor sells pretzels for $1.25 each and hotdogs for $2.00 each. One day he sold 82 items for a total of $134.75. How many pretzels did he sell?

3. If $\frac{A+1}{B-1}=\frac{5}{6}$ and $\frac{A-1}{B-1}=\frac{3}{4}$, then find the value of $A + B$.

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #4, February 2004

Answers

1. 12

2. 39

3. 44

1. Since Julia and Michelle are running toward each other, they are closing the four-mile gap between them at the sum of their speeds, which is 6 + 14 = 20 miles per hour. Using the formula $D = R \times T$, we have the value of *D* and *R*, so we can solve for *T*.
$4 = 20 \times T$, so $T = \frac{4}{20} = \frac{1}{5}$ hour or 60 ÷ 5 = **12** minutes.

2. If the vendor had sold the pretzels for the same price as the hotdogs, he would have taken in 82 × \$2.00 = \$164. Since the pretzels actually sold for \$0.75 less than the hotdog, the difference \$164 – \$134.75 = \$29.25 can be thought of as the number of pretzels times the \$0.75 that he did not take in for each pretzel. Since \$29.25 ÷ \$0.75 = 39, he must have sold **39** pretzels. Most people would not think so hard about it and would just trust the algebra. Let *D* be the number of hotdogs and let *P* be the number of pretzels. We can write the following two equations from the information given:

$$D + P = 82$$
$$2D + 1.25P = 134.75$$

Doubling the first equation and subtracting the second equation from it, we get:

$$\begin{array}{r} 2D \;\;+\;\; 2P = 164 \\ 2D + 1.25P = 134.75 \\ \hline 0.75P = 29.25 \end{array}$$

Dividing both sides of the last equation by 0.75, we get *P* = **39**.

3. Find the cross product of each proportion and make a system of equations:
$\begin{cases} 6(A+1) = 5(B-1) \\ 4(A-1) = 3(B-1) \end{cases} \Rightarrow \begin{cases} 6A + 6 = 5B - 5 \\ 4A - 4 = 3B - 3 \end{cases} \Rightarrow \begin{cases} 6A - 5B = -11 \\ 4A - 3B = 1 \end{cases}$. Doubling the top equation and tripling the bottom, we get: $\begin{cases} 12A - 10B = -22 \\ 12A - 9B = 3 \end{cases}$. Subtracting the bottom equation from the top, we get $-B = -25$, so *B* = 25. Substituting this value for *B* in the equation $6A - 5B = -11$, we get $6A - 125 = -11 \Rightarrow 6A = 114 \Rightarrow$ $A = 114 \div 6 = 19$. Finally, we can compute that *A* + *B* = 19 + 25 = **44**.

You may use a calculator

Category 6
Team Questions
Meet #4, February 2004

1. The Martian day is about 24 hours and 40 minutes long. Let's assume that on February 24th local noon for the rover Opportunity on Mars is at the same time as local noon for the controllers on earth (in California). How many Earth days will pass before these local noons coincide again?

2. Consider the arithmetic sequence 5, 14, 23, 32, etc., which is increasing by 9. Consider also the arithmetic sequence 2, 13, 24, 35, etc., which is increasing by 11. What is the average of the first two numbers that appear in both sequences? Express your answer as a decimal to the nearest tenth.

3. What integer can be added to both the numerator and denominator of the fraction $\frac{5}{8}$ to create a fraction that has a value of $\frac{5}{6}$?

4. Coach McMod gave his twelve basketball players jerseys with the numbers 0 through 11 and had them circle up so that their jersey numbers would be in clockwise ascending order. He told the team that the day's passing pattern would be the counting numbers. The first player would pass 1 to the left, the next player would pass 2 to the left, then 3, etc. He gave the ball to the person wearing number 1 and blew the whistle to begin. The passing went on for a long time. What is the sum of the jersey numbers of the players who never touched the ball?

5. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the sum of all the n from set A that satisfy the congruence $n^5 \equiv 1 \pmod{11}$. In other words, find the sum of all the n from set A whose fifth power is equal to 1 in mod 11.

Answers

1. ______________ = A
2. ______________ = B
3. ______________ = C
4. ______________ = D
5. ______________ = E
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression. Express your result as a ***common fraction***, not a mixed number or a decimal.

$$\frac{\sqrt{\dfrac{BC}{A+C}}}{D+E}$$

Solutions to Category 6
Team Questions
Meet #4, February 2004

Answers

1. 37

2. 117.5

3. 10

4. 18

5. 22

6. $\frac{1}{8}$

1. There are 36 time periods of 40 minutes in 24 hours. This means that after 36 Martian rotations the local noons will coincide again. The hard part is understanding why **37** days will have passed on Earth. Suppose person *A* runs more slowly than person *B*. If it takes person *A* $\frac{3}{2}$ the amount of time to go the same distance as person *B*, then person *A* will cover $\frac{2}{3}$ the distance as person *B* in the same amount of time. It takes Mars $24\frac{2}{3}$ hours compared to Earth's 24 hours to complete one rotation, so Mars spins more slowly. This ratio can be reduced as follows: $\frac{24\frac{2}{3}\text{ hours}}{24\text{ hours}} = \frac{\frac{74}{3}}{\frac{72}{3}} = \frac{74}{72} = \frac{37}{36}$.
Since Mars takes $\frac{37}{36}$ the amount of time to go the same distance (one rotation), it will go $\frac{36}{37}$ the distance in the same amount of time. Thus, Mars will spin 36 times and Earth **37** times.

2. If we can find the first number that appears on both lists, the next one will just be 99 greater, since 99 is the least common multiple of 9 and 11.

5, 14, 23, 32, 41, 50, 59, 68, ..., 167
2, 13, 24, 35, 46, 57, 68, ..., 167

The average of 68 and 167 is $\frac{68+167}{2} = \frac{235}{2} = \mathbf{117.5}$.

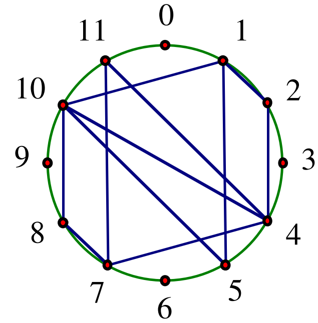
3. We can set up a proportion, take the cross product, and then solve for *n* as follows:

$$\frac{5+n}{8+n} = \frac{5}{6} \Rightarrow 6(5+n) = 5(8+n) \Rightarrow 30+6n = 40+5n \Rightarrow n = 10$$

We should check this by actually adding 10 to both numerator and denominator of $\frac{5}{8}$ and see if we get a fraction with a value of $\frac{5}{6}$.

$$\frac{5+10}{8+10} = \frac{15}{18} = \frac{5}{6}$$

4. The figure at right shows the path of the ball. The order by jersey number is: 1, 2, 4, 7, 11, 4, 10, 5, 1, 10, 8, 7, 7, 8, 10, 1, 5, 10, 4, 11, 7, 4, 2, 1, etc. The players with jersey numbers 0, 3, 6, and 9 never touched the ball. The desired sum is 0 + 3 + 6 + 9 = **18**.



5. The table below shows the mod 11 powers of all the elements of set *A*. We wish to know which bases (on the left) give a value of 1 in the fifth column. They are 1, 3, 4, 5, and 9. Their sum is **22**. At the right of the table, we see the base ten values of the fifth powers with their mod 11 equivalents. (Reminder: To find the mod 11 value, divide these numbers by 11 and look at the remainder.)

Exponent

Base	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	5	10	9	7	3	6	1
3	3	9	5	4	1	3	9	5	4	1
4	4	5	9	3	1	4	5	9	3	1
5	5	3	4	9	1	5	3	4	9	1
6	6	3	7	9	10	5	8	4	2	1
7	7	5	2	3	10	4	6	9	8	1
8	8	9	6	4	10	3	2	5	7	1
9	9	4	3	5	1	9	4	3	5	1
10	10	1	10	1	10	1	10	1	10	1

$$2^5 = 32 \equiv 10 (\text{Mod } 11)$$
$$3^5 = 243 \equiv 1 (\text{Mod } 11)$$
$$4^5 = 1024 \equiv 1 (\text{Mod } 11)$$
$$5^5 = 3125 \equiv 1 (\text{Mod } 11)$$
$$6^5 = 7776 \equiv 10 (\text{Mod } 11)$$
$$7^5 = 16807 \equiv 10 (\text{Mod } 11)$$
$$8^5 = 32768 \equiv 10 (\text{Mod } 11)$$
$$9^5 = 59049 \equiv 1 (\text{Mod } 11)$$
$$10^5 = 100000 \equiv 10 (\text{Mod } 11)$$

6. Substituting the correct values for *A* through *E*, we get:

$$\frac{\sqrt{\frac{BC}{A+C}}}{D+E} = \frac{\sqrt{\frac{117.5 \cdot 10}{37+10}}}{18+22} = \frac{\sqrt{\frac{1175}{47}}}{40} = \frac{\sqrt{25}}{40} = \frac{5}{40} = \mathbf{\frac{1}{8}}$$