Meet #3
January 2004

Intermediate Mathematics League of Eastern Massachusetts

Meet #3
January 2004

Category 1
Mystery
Meet #3, January 2004

1. A certain auditorium has six seats arranged and numbered as suggested by the table below which shows the first four rows.

Row 4	25	26	27	28	29	30	31	32	33	34	35	36
Row 3		15	16	17	18	19	20	21	22	23	24	
Row 2			7	8	9	10	11	12	13	14		
Row 1				1	2	3	4	5	6			

If the pattern continues, what is the number of the seat directly behind seat number 60? (Note: "Behind" implies that the row number is greater.)

2. Find the least whole number that satisfies the following conditions:

Its digits are only 3's and 4's.
It contains at least one 3 and at least one 4.
It is divisible by both 3 and 4.

3. A certain function takes any number of inputs and finds the sum of all the products that are made by multiplying the elements in all the possible subsets of the inputs. Let's call this function the "prosum" function and put the inputs in brackets as shown below. The general form of the prosum function with three inputs would look like this:

$$\text{prosum}[a,b,c] = abc + ab + ac + bc + a + b + c$$

Find the value of $\text{prosum}[2,4,6]$.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1
Mystery
Meet #3, January 2004

Answers

1. 77

2. 3444

3. 104

1. Each new row of seats has two more seats than the previous row. The last seat on the right has the highest number in each row. Let's keep track of these numbers.

Row 1 ends in 6.
Row 2 ends in 6 + 8 = 14.
Row 3 ends in 14 + 10 = 24.
Row 4 ends in 24 + 12 = 36.
Row 5 ends in 36 + 14 = 50.
Row 6 ends in 50 + 16 = 66.
Row 7 ends in 66 + 18 = 84.

Now let's draw part of the sixth and seventh rows:

Row	...	7	7	8	8	8	3	8	8
7		7	9	0	1	2	4	3	4
Row	...	6	6	6	6	6	6	6	
6		0	1	2	3	4	5	6	

Seat number **77** is directly behind seat number 60.

2. In order for the number to be divisible by 3, we must use three 4's along with the one 3 that is required. Since 34 is not divisible by 4, our number must end in 44, which is divisible by 4. Our options are now 3444 and 4344, and **3444** is the lesser of the two.

3. The evaluation of $\text{prosum}[2,4,6]$ is shown below.

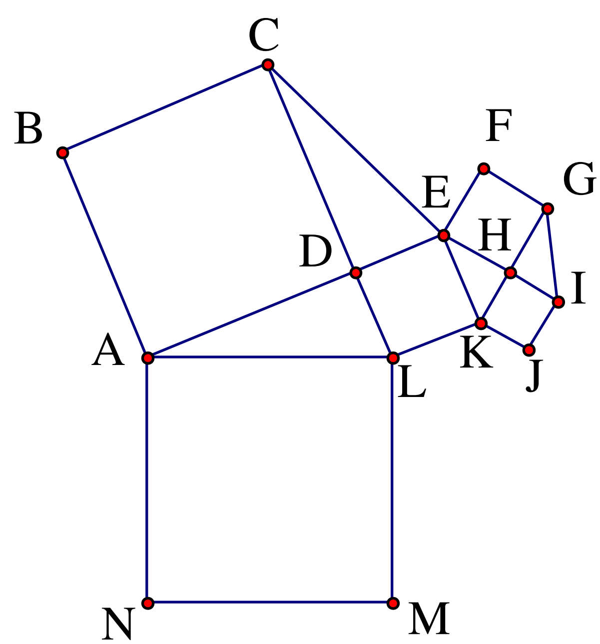
$$\begin{aligned}\text{prosum}[2,4,6] &= 2\cdot 4\cdot 6 + 2\cdot 4 + 2\cdot 6 + 4\cdot 6 + 2 + 4 + 6\\ &= 48 + 8 + 12 + 24 + 2 + 4 + 6\\ &= 104\end{aligned}$$

Category 2
Geometry
Meet #3, January 2004

1. How many diagonals can be drawn in a decagon?
Note: A decagon is a polygon with 10 sides and a diagonal of a polygon is a segment which connects any two non-consecutive vertices.

2. A regular polygon has an interior angle measure that is greater than 144 degrees and less than 150 degrees. How many sides does the polygon have?

3. In the figure at right, every quadrilateral is a square and every triangle is a right triangle. The area of square ABCD is 144 square units and the area of square EFGH is 16 square units. Length LK is 5 units. How many units are in the perimeter of the figure (polygon ABCEFGIJKLMN)?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 2
Geometry
Meet #3, January 2004

Answers

1. 35

2. 11

3. 100

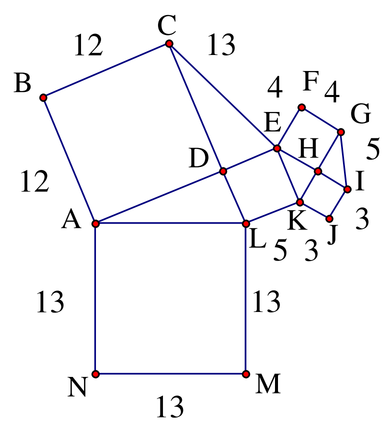
1. From each of the ten vertices of a decagon, one can draw seven diagonals. If we multiply 7 by 10 to get 70, we will have counted every diagonal twice. Therefore there must be 70 ÷ 2 = **35** diagonals in a decagon.

2. A polygon with n sides can be subdivided into $n - 2$ triangles, each with an angle sum of 180. This gives the n-gon a total angle sum of $180(n-2)$ or $180n-360$. If the n-gon is regular, then all n of its interior angles will have the angle measure $(180n-360)/n$. We need to solve the double inequality $144 < (180n-360)/n < 150$ for a whole-number value of n.

$144 < (180n-360)/n$	$(180n-360)/n < 150$
$144n < 180n - 360$	$180n - 360 < 150n$
$360 < 180n - 144n$	$180n - 150n < 360$
$360 < 36n$	$30n < 360$
$10 < n$	$n < 12$

The only whole number value between 10 and 12 is 11, so the polygon must have **11** sides.

3. Square ABCD has an area of 144 square units, so its side length is 12. Length DL is 5, so triangle ADL is a 5-12-13 triangle as the Pythagorean Theorem confirms: $5^2+12^2=25+144=169$ and $13^2=169$. We are also given that square EFGH has an area of 16 square units, so its side length is 4 units. Triangle EHK is a 3-4-5 triangle, since $5^2-4^2=25-16=9$ and $3^2=9$. We can now label all the side lengths in the figure as shown at right. The perimeter is $12+12+13+4+4+5+3+3+5+13+13+13$ = **100** units.



Category 3
Number Theory
Meet #3, January 2004

1. Solve the following base five equation for n. Express your result in base five.

$$2343_{\text{base five}} + n_{\text{base five}} = 4032_{\text{base five}}$$

2. A box of twelve eggs is commonly refered to as a "dozen" eggs. Less common are the words "gross", which means a dozen dozen, and "great gross", which means a dozen dozen dozen. How many eggs are there in two great gross five gross nine dozen three?

3. Evaluate the following expression. Write your result in scientific notation.

$$\frac{\left(3.8 \times 10^{17}\right)\left(1.2 \times 10^{-4}\right)}{\left(5.7 \times 10^{-13}\right)}$$

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #3, January 2004

Answers

1. 1134

2. 4287

3. 8×10^{25}

1. To find the base five value of n that satisfies the equation $2343_{\text{base five}} + n_{\text{base five}} = 4032_{\text{base five}}$, we need to subtract 2343 from 4032 in base five. When we "borrow", we need to remember that we get five instead of ten.

The computation shown at right does not show evidence of the borrowing that most people do to get this result. For example, that first 2 minus 3 in the ones place would become 7 minus 3 when we borrow a set of five from the column to the left. The 3 in the second column would become a 2 and then we would have to borrow again.

$$\begin{array}{r} 4032 \\ \underline{-2343} \\ 1134 \end{array}$$

We should confirm that n is **1134** by returning to the original equation and checking the addition.

$$2343_{\text{base five}} + 1134_{\text{base five}} = 4032_{\text{base five}}$$

2. Before we can interpret the phrase "two great gross five gross nine dozen three", we need to establish the base ten values of these base twelve words. Dozen means 12, gross means 12 × 12 = 144, and great gross means 12 × 12 × 12 = 144 × 12 = 1728. Now we can translate the word expression to the numerical expression $2 \times 1728 + 5 \times 144 + 9 \times 12 + 3$ and evaluate it. We get $2 \times 1728 + 5 \times 144 + 9 \times 12 + 3 = 3456 + 720 + 108 + 3 = \mathbf{4287}$ eggs.

3. Since the expression consists of multiplication and division, we can rearrange them as follows: $\dfrac{(3.8 \times 10^{17})(1.2 \times 10^{-4})}{(5.7 \times 10^{-13})} = \dfrac{3.8 \times 1.2}{5.7} \times \dfrac{10^{17} \times 10^{-4}}{10^{-13}}$. The number part can be reduced to $\dfrac{2 \times 1.9 \times 1.2}{3 \times 1.9} = \dfrac{2 \times 1.2}{3} = 0.8$ and the power of ten becomes $\dfrac{10^{17} \times 10^{-4}}{10^{-13}} = 10^{17-4-(-13)} = 10^{26}$. We now have $0.8 \times 10^{26} = \mathbf{8 \times 10^{25}}$.

Category 4
Arithmetic
Meet #3, January 2004

1. Evaluate the following expression. Write your result as a mixed number with the fraction part in lowest terms.

$$\left(5^{7}\right)\bullet\left(7^{-3}\right)\bullet\left(15^{-5}\right)\bullet\left(21^{3}\right)\bullet(-1)^{9}$$

2. Evaluate the following expression.

$$\sqrt[3]{\sqrt[6]{64}+\sqrt[4]{256}+\sqrt[2]{36}+\sqrt[5]{243}+\sqrt[3]{125}+\sqrt[3]{343}}$$

3. How many whole numbers are between $\left(\sqrt{\frac{81}{100}}\right)^{-1}$ and $\left(\sqrt[3]{\frac{125}{1000}}\right)^{-2}$?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 4
Arithmetic
Meet #3, January 2004

Answers

1. $-2\frac{7}{9}$

2. 3

3. 2

1. First of all, let's take care of the sign of our result. Since a negative times a negative is a positive, raising –1 to an even power gives positive one and raising –1 to an odd power gives negative one. Our answer will be negative. It might help to rewrite each of the composite bases as products of their prime factors.

$15^{-5} = (3 \bullet 5)^{-5} = 3^{-5} \bullet 5^{-5}$ and $21^3 = (3 \bullet 7)^3 = 3^3 \bullet 7^3$

We can now rewrite the entire expression as follows:

$$(5^7) \bullet (7^{-3}) \bullet (15^{-5}) \bullet (21^3) = (5^7) \bullet (7^{-3}) \bullet (3^{-5} \bullet 5^{-5}) \bullet (3^3 \bullet 7^3)$$

Now we can collect like bases and combine exponents.

$$3^{-5+3} \bullet 5^{7-5} \bullet 7^{-3+3} = 3^{-2} \bullet 5^2 = \frac{5^2}{3^2} = \frac{25}{9}$$

Remember that our answer is negative and we must express it as a mixed number. Thus we have $-\frac{25}{9} = \mathbf{-2\frac{7}{9}}$

2. First let's consider each of the separate radicals inside the larger radical.

$\sqrt[6]{64} = 2$ since $2^6 = 64$. $\sqrt[4]{256} = 4$ since $4^4 = 256$. $\sqrt[2]{36} = 6$ since $6^2 = 36$.
$\sqrt[5]{243} = 3$ since $3^5 = 243$. $\sqrt[3]{125} = 5$ since $5^3 = 125$. $\sqrt[3]{343} = 7$ since $7^3 = 343$.

Now we can simplify as follows:

$$\sqrt[3]{\sqrt[6]{64} + \sqrt[4]{256} + \sqrt[2]{36} + \sqrt[5]{243} + \sqrt[3]{125} + \sqrt[3]{343}} = \sqrt[3]{2+4+6+3+5+7} = \sqrt[3]{27} = \mathbf{3}$$

3. The value of $\left(\sqrt{\frac{81}{100}}\right)^{-1}$ is $\left(\frac{9}{10}\right)^{-1} = \frac{10}{9} = 1\frac{1}{9}$ and the value of $\left(\sqrt[3]{\frac{125}{1000}}\right)^{-2}$ is $\left(\frac{5}{10}\right)^{-2} = \left(\frac{10}{5}\right)^2 = \frac{100}{25} = 4$. There are two (**2**) whole numbers between $1\frac{1}{9}$ and 4, namely 2 and 3.

Category 5
Algebra
Meet #3, January 2004

1. What is the positive difference between the two solutions of the following equation? Express your answer as a mixed number in lowest terms.

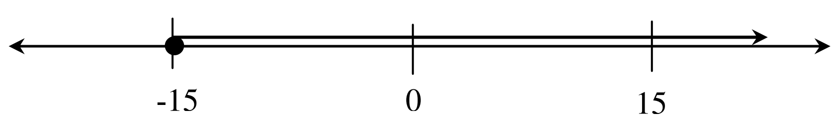
$$\left|\frac{3x-7}{5}-8\right|=4$$

2. What is the sum of all the integer solutions of the double inequality below?

$$-3 \le \frac{x+6}{14} \le 4$$

3. What value of *B* will make the solution of the inequality below match the solution given by the graph below?

$$\frac{-x+B}{17} \le 3$$



Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #3, January 2004

Answers

1. $13\frac{1}{3}$

2. 99

3. 36

1. We must solve the two distinct equations below:

$$\frac{3x-7}{5}-8=4 \qquad\qquad \frac{3x-7}{5}-8=-4$$
$$\frac{3x-7}{5}=12 \qquad\qquad \frac{3x-7}{5}=4$$
$$3x-7=60 \qquad\qquad 3x-7=20$$
$$3x=67 \qquad\qquad 3x=27$$
$$x=\frac{67}{3} \qquad\qquad x=\frac{27}{3}$$
$$x=22\frac{1}{3} \qquad\qquad x=9$$

The desired difference is $22\frac{1}{3}-9=\mathbf{13\frac{1}{3}}$.

2. First, we need to solve the two inequalities separately.

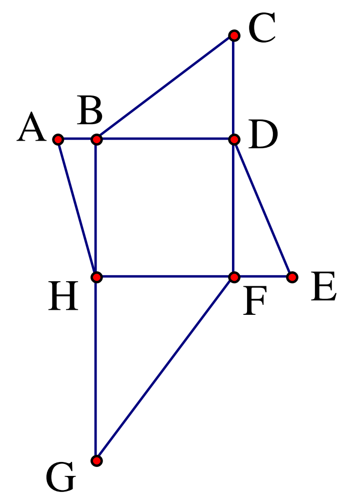
$$-3\le\frac{x+6}{14} \qquad\qquad \frac{x+6}{14}\le 4$$
$$-42\le x+6 \qquad\qquad x+6\le 56$$
$$-48\le x \qquad\qquad x\le 50$$

We now need to find the sum of the integers from –48 to 50. Every negative integer in this list will find its opposite and make a sum of zero, leaving just 49 and 50. The final sum is **99**.

3. The graph shows the solution $x\ge -15$. We need to isolate the variable *x* in the inequality. $\frac{-x+B}{17}\le 3 \Rightarrow -x+B\le 51 \Rightarrow -x\le 51-B$. Multiplying both sides by –1 switches the direction of the inequality, so we end up with: $x\ge -(51-B)$. Now we need to solve $-(51-B)=-15$ for B. This simplifies to 51 – *B* = 15, and finally *B* = **36**.

Category 6
Team Questions
Meet #3, January 2004

1. Find the base five value of 2^{22}.
(Note: The exponent 22 is in base five as well.)



2. Square BDFH has side length 24 units. $m\overline{AH} = 25$, $m\overline{BC} = 30$, $m\overline{DE} = 26$, and $m\overline{FG} = 40$ units. How many square units are in the area of polygon ABCDEFGH?

3. Solve the following base eight equation for n.
Express the value of n in base eight.

$$4(10n + 13)_{\text{base eight}} = 13014_{\text{base eight}}$$

4. Recall the "prosum" function that finds the sum of all the products that are made by multiplying the elements in all the possible subsets of the inputs. The general form of the equation with three inputs is:

$$\text{prosum}[a,b,c] = abc + ab + ac + bc + a + b + c.$$

Extend the function to four inputs and find the value of $\text{prosum}[2,3,4,5]$.

5. A sequence of numbers is called a Fibonacci-type Sequence if each number (after the first two) is the sum of the two numbers that preceed it. If the numbers 83, x, y, 323 are four consecutive numbers in a Fibonacci-type Sequence, then find the value of y.

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. ________________

6. Evaluate the following expression using the values obtained in questions 1 through 5. Write your result as a mixed number in lowest terms.
Note: For the purpose of this problem, we will treat all the values as base ten numbers.

$$\frac{\sqrt{\dfrac{A-(D+E)+1}{B}}}{\sqrt[4]{C-1}}$$

Solutions to Category 6
Team Questions
Meet #3, January 2004

Answers

1. 112341

2. 1380

3. 257

4. 359

5. 203

6. $2\frac{1}{4}$

1. One way to tackle this problem is to find the base ten value of the base five power of two and then convert it back to base five. The 22 in the exponent means two 5's and 2, which is 12 in base ten. Thus we have:
$2^{22}_{\text{base five}} = 2^{12}_{\text{base ten}} = 4096_{\text{base ten}}$. Now, in order to convert this base ten number to base five, we will need to know the place values in base five. In reverse order, they are: 1, 5, 25, 125, 625, 3125, etc. Subtracting 3125 from 4096 just once, we find that we need a 1 in the sixth place value from the right and we have 971 more to deal with. We then subtract as many 625's as we can, etc. The equation below summarizes:

$$4096 = \mathbf{1}\times 3125 + \mathbf{1}\times 625 + \mathbf{2}\times 125 + \mathbf{3}\times 25 + \mathbf{4}\times 5 + \mathbf{1}\times 1$$

Each product on the right side of the equation contains the base-five digit (in bold) times the base-five place value. Our base-five number is thus **112341**.

2. The area of square BDFH itself is $24 \times 24 = 576$ square units. As for the right triangles, they all have one leg of 24 units. We are given the lengths of the hypotenuses and we will use the Pythagorean Theorem to calculate the length of the other legs. Once we have both legs, we can calculate the area of each triangle.

For triangle *ABH*, we have $m\overline{AH} = 25$, so $25^2 - 24^2 = 625 - 576 = 49 = 7^2$. This means $m\overline{AB} = 7$ and the area of triangle *ABH* is $\frac{1}{2}\times 24\times 7 = 84$ square units.

For triangle *BCD*, we have $m\overline{BC} = 30$, so $30^2 - 24^2 = 900 - 576 = 324 = 18^2$. This means $m\overline{CD} = 18$ and the area of triangle *BCD* is $\frac{1}{2}\times 24\times 18 = 216$ square units.

For triangle *DEF*, we have $m\overline{DE} = 26$, so $26^2 - 24^2 = 676 - 576 = 100 = 10^2$. This means $m\overline{EF} = 10$ and the area of triangle *DEF* is $\frac{1}{2}\times 24\times 10 = 120$ square units.

For triangle *FGH*, we have $m\overline{FG} = 40$, so $40^2 - 24^2 = 1600 - 576 = 1024 = 32^2$. This means $m\overline{GH} = 32$ and the area of triangle *FGH* is $\frac{1}{2}\times 24\times 32 = 384$ square units.

The total area of the figure is the area of the square plus the areas of the four triangles and that is $576 + 84 + 216 + 120 + 384 = \mathbf{1380}$ square units.

3. Using the same reasoning that we use to solve equations in base ten, we might divide both sides by 4 first or distribute the 4 first. Either way, we must remember that we are in base eight.

$$4(10n+13)_{\text{base eight}} = 13014_{\text{base eight}}$$
$$10n+13 = 2603$$
$$10n = 2570$$
$$n = \mathbf{257}_{\text{base eight}}$$

4. The evaluation of prosum[2,3,4,5] is shown below.

$$\begin{aligned}\text{prosum}[2,3,4,5] &= 2\cdot3\cdot4\cdot5 + 2\cdot3\cdot4 + 2\cdot3\cdot5 + 2\cdot4\cdot5 + 3\cdot4\cdot5 + 2\cdot3 + 2\cdot4 + 2\cdot5 \\ &\quad + 3\cdot4 + 3\cdot5 + 4\cdot5 + 2 + 3 + 4 + 5 \\ &= 120 + 24 + 30 + 40 + 60 + 6 + 8 + 10 + 12 + 15 + 20 + 2 + 3 + 4 + 5 \\ &= \mathbf{359}\end{aligned}$$

5. Using the rule of formation for the next term in a Fibonacci-type Sequence, we can write the following two equations: $83 + x = y$ and $x + y = 323$. Substituting the value $83 + x$ in place of the *y* in the second equation, we get: $x + (x + 83) = 323$. We now have one equation with one unknown and we can solve for *x* as follows: $2x + 83 = 323 \Rightarrow 2x = 240 \Rightarrow x = 120$. Finally, we can find the value of *y* by substituting this value of *x* into the first equation. We get: $y = 83 + 120 = \mathbf{203}$

6. Substituting the values for *A* through *E* into the expression, we get:

$$\frac{\sqrt{\frac{A-(D+E)+1}{B}}}{\sqrt[4]{C-1}} = \frac{\sqrt{\frac{112341-(359+203)+1}{1380}}}{\sqrt[4]{257-1}} = \frac{\sqrt{\frac{112341-(562)+1}{1380}}}{\sqrt[4]{256}}$$

$$= \frac{\sqrt{\frac{111779+1}{1380}}}{4} = \frac{\sqrt{\frac{111780}{1380}}}{4} = \frac{\sqrt{\frac{11178}{138}}}{4} = \frac{\sqrt{81}}{4} = \frac{9}{4} = \mathbf{2\frac{1}{4}}$$