

Intermediate
Mathematics League
of
Eastern Massachusetts

Meet #2
November 2003

Category 1

Mystery

Meet #2, November 2003

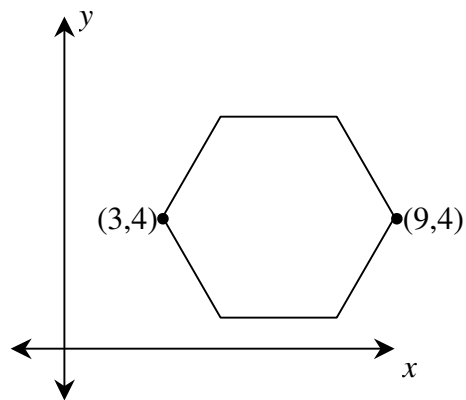
1. In the following shape sentences, same shapes have the same numbers and different shapes have different numbers. Find the value of the circle.

$$\begin{array}{r} \triangle + \square + \square = 14 \\ \triangle + \bigcirc - \square = 13 \\ \triangle + \bigcirc = 19 \end{array}$$

2. Together Albert and Brendan weigh 121 pounds.
Brendan and Camille weigh 113 pounds together.
Camille and Dana together weigh 83 pounds.
How many pounds do Albert and Dana weigh together?

3. A regular hexagon lives in the first quadrant of a Cartesian Coordinate System as shown below. How many units are in the perimeter of the hexagon?

Answers	
1.	_____
2.	_____
3.	_____



Solutions to Category 1

Mystery

Meet #2, November 2003

Answers

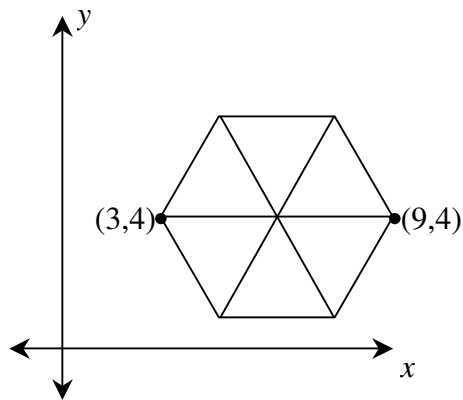
1. 17

2. 91

3. 18

1. The third shape sentence shows that a triangle and a circle together have a value of 19. The second shape sentence has the same triangle and circle, but the value of the square has been subtracted, reducing the 19 to 13. The value of the square must be 6. Writing the value 6 in both squares of the first shape sentence, it becomes clear that the triangle must have a value of 2. Finally, returning to the third shape sentence, we see that the circle must have a value of **17**.

2. If we combine the first and the third sentences, we will know that Albert and Brendan with Camille and Dana must weigh 121 plus 83 or 204 pounds. We know from the second sentence that the combined weight of Brendan and Camille is 113 pounds. If we subtract this 113 from 204, we get **91** pounds, which must be the combined weight of Albert and Dana.



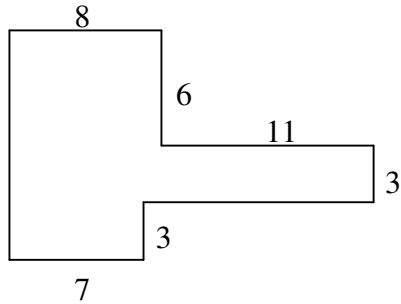
3. A regular hexagon can be subdivided into 6 equilateral triangles as shown in the figure below. Since the horizontal distance between (3,4) and (9,4) is $9 - 3 = 6$ units, the side length of the equilateral triangles must be $6 \div 2 = 3$ units. Thus the perimeter of the hexagon is $3 \times 6 = \mathbf{18}$ units.

Category 2

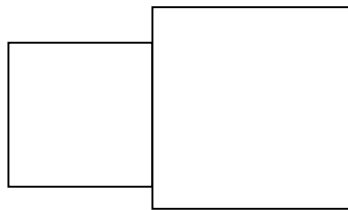
Geometry

Meet #2, November 2003

1. Find the number of feet in the perimeter of the figure below. All angles are right angles and all lengths are in feet.

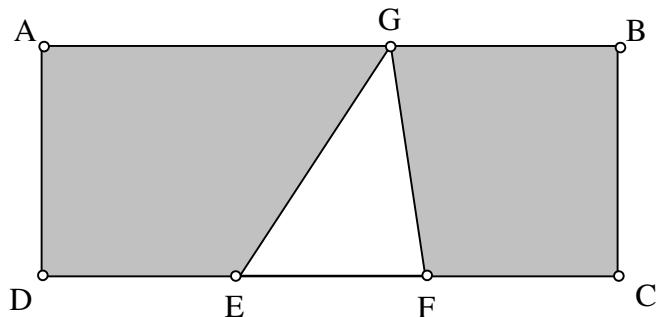


2. The two squares shown side by side in the figure below have a total area of 74 square inches. The side length of each square is a whole number of inches. How many inches are in the perimeter of the figure? (Note: The line where the two squares touch is not to be counted in the perimeter of the figure.)



3. In the figure below, $ABCD$ is a rectangle. The points E and F trisect side \overline{DC} , meaning they cut it in three equal pieces. G is a point on side \overline{AB} . The length of side \overline{AB} is 3 inches and the length of \overline{BC} is 1.2 inches. Find the number of square inches in the shaded area of the figure.

Answers	
1.	_____
2.	_____
3.	_____



Solutions to Category 2
 Geometry
 Meet #2, November 2003

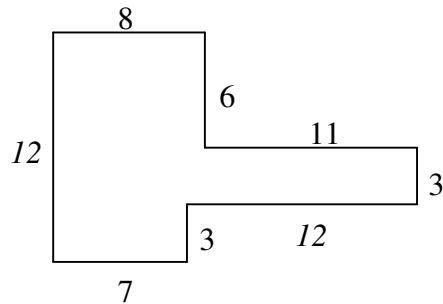
Answers

1. 62

2. 38

3. 3

1. The unmarked height on the left side of the figure is the sum of the three vertical lengths on the right, namely $6 + 3 + 3 = 12$ feet. Similarly, the other unmarked length must be $8 + 11 - 7 = 12$ feet also. The perimeter of the figure is thus: $8 + 6 + 11 + 3 + 12 + 3 + 7 + 12 = 62$ feet. Alternatively, one could reason that there must be two vertical totals of 12 feet and two horizontal totals of 19 feet, for a perimeter of: $2 \times 12 + 2 \times 19 = 24 + 38 = 62$ feet.



2. The two perfect squares with a sum of 74 must be 49 and 25. This means the side length of the smaller square is 5 inches and the side length of the larger square must be 7 inches. Although there are two unknown lengths where the smaller square meets the larger square (call one of them x and the other y), the total height $x + 5 + y$ must equal 7. Thus the perimeter of the figure is $4 \times 7 + 2 \times 5 = 28 + 10 = 38$.

3. The area of rectangle $ABCD$ is $1.2 \times 3 = 3.6$ square inches. Since E and F trisect side \overline{DC} , the length of segment \overline{EF} must be $3 \div 3 = 1$ inch. Also, regardless of where point G is on \overline{AB} , the height of triangle EFG is 1.2 inches. Thus the area of triangle EFG is $\frac{1}{2} \times 1 \times 1.2 = 0.6$ square inches. Subtracting the area of triangle EFG from the area of rectangle $ABCD$, we get $3.6 - 0.6 = 3$ square inches, which is the area of the shaded regions.

Category 3
Number Theory
Meet #2, November 2003

1. The product of two whole numbers is 56 and their GCF (greatest common factor) is 1. If both numbers are less than 56, what is the positive difference between these two numbers?

2. The GCF of n and m is 99 and the LCM (least common multiple) is 594. If n is 198, then what is the value of m ?

3. The product of two whole numbers, a and b , is 330. The positive difference between a and b is a prime number. The sum of the digits of this difference is also a prime number. What is the LCM of a and b ?

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 3
Number Theory
Meet #2, November 2003

Answers

1. 1

2. 297

3. 330

1. All possible factor pairs of 56 are listed below:

$$1 \times 56$$

$$2 \times 28$$

$$4 \times 14$$

$$7 \times 8$$

The GCF of 2 and 28 is 2 and the GCF of 4 and 14 is also 2. The pair 7 and 8 have a GCF of 1 and they are both less than 56. The positive difference between these numbers is **1**.

2. The prime factorization of 594, the LCM of n and m , is $2^1 \times 3^3 \times 11^1$ and the prime factorization of 99, the GCF of n and m , is $3^2 \times 11^1$. Both n and m must be multiples of 99, their GCF, and we must account for all the prime factors of the LCM. If n is 198, which is 2×99 , then m must be 3×99 , which is **297**, to account for the extra factor of 3 in the LCM.

3. The prime factorization of 330 is $2 \times 3 \times 5 \times 11$. Since there is only one of each prime factor of 330, any two whole numbers, a and b , that have a product of 330 must be relatively prime, meaning their GCF is sure to be 1. For any two numbers that are relatively prime, their LCM must equal their product. This means that we do not actually have to determine the values of a and b and the other clues are not needed. The LCM of a and b is **330**.

Category 4
Arithmetic
Meet #2, November 2003

1. What is $87\frac{1}{2}\%$ of $\frac{5}{7}$ of 25% of $\frac{4}{9}$ of 720?

2. What number is $83\frac{1}{3}\%$ greater than the repeating decimal $0.\overline{54}$? Express your answer in simplest terms.

3. During the 48 hours from 9:00 PM on Friday to 9:00 PM on Sunday, Laura spent $\frac{1}{3}$ of her time sleeping, $\frac{1}{8}$ of her time raking leaves, $\frac{1}{12}$ of her time reading a book, $\frac{1}{16}$ of her time working on her science project, $\frac{1}{24}$ of her time riding in the car, and $\frac{1}{48}$ of her time watching television. How many hours in this time period remained for all other activities?

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
 Arithmetic
 Meet #2, November 2003

Answers

1. 50

1. First, we might want to convert the percentages into fractions:

$$87\frac{1}{2}\% = \frac{7}{8} \text{ and } 25\% = \frac{1}{4}.$$

2. 1

The word *of* means multiply, so we can translate the English to the following numerical expression and cancel common factors to find the final product:

3. 16

$$\frac{7}{8} \times \frac{5}{7} \times \frac{1}{4} \times \frac{4}{9} \times \frac{720}{1} = \frac{5 \times 720}{8 \times 9} = \frac{5 \times 72 \times 10}{72} = 5 \times 10 = \mathbf{50}$$

2. Suppose $x = 0.\overline{54}$, then $100x = 54.\overline{54}$. Subtracting the first equation from the second, we get:

$$\begin{array}{r} 100x = 54.\overline{54} \\ - \quad 1x = 0.\overline{54} \\ \hline 99x = 54 \end{array}$$

Solving for x , we get: $x = \frac{54}{99} = \frac{6}{11}$. An increase of $83\frac{1}{3}\%$ will include all 100% of the original, so the new quantity will be $183\frac{1}{3}\%$, or $1\frac{5}{6}$, of the original amount, which in this case is $\frac{6}{11}$. Multiplying these two quantities, we get:

$$1\frac{5}{6} \times \frac{6}{11} = \frac{11}{6} \times \frac{6}{11} = \frac{66}{66} = \mathbf{1}$$

3. Laura spent 16 hours sleeping, 6 hours raking, 4 hours reading, 3 hours working on her science project, 2 hours riding in the car, and 1 hour watching television. That accounts for $16 + 6 + 4 + 3 + 2 + 1 = 32$ hours. She still had $48 - 32 = \mathbf{16}$ hours for all other activities.

Category 5

Algebra

Meet #2, November 2003

1. Pick's formula gives the area of any polygon whose vertices are on the lattice points of a square grid. Such polygons can be made with rubber-bands on a peg board such as the Geoboard. Pick's formula can be stated as follows:

$$A = \frac{1}{2}B + I - 1$$

where A is the area, B is the number of pegs touching the border of the figure (at vertices and along sides), and I is the number of pegs in the interior of the polygon. A certain polygon is made with rubber bands on a peg board and has an area of 70.5 square units. If there are 60 pegs in the interior of the polygon, how many pegs must be on the border of the figure?

2. Four times the sum of a number and two is five more than seven times an amount six less than the number. Find the number.

3. The sum of three consecutive multiples of 120 is the same as the sum of five consecutive multiples of 96. If the least of the three multiples of 120 is 840, then what is the value of the greatest of the multiples of 96?

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 5
Algebra
Meet #2, November 2003

Answers

1. 23
2. 15
3. 768

1. Substituting the value 70.5 for A and 60 for B in Pick's formula, we get:

$$70.5 = \frac{1}{2}B + 60 - 1 \text{ or } 70.5 = \frac{1}{2}B + 59$$

Solving for B , we get $11.5 = \frac{1}{2}B$, then $23 = B$.

There must be **23** pegs on the border of the figure.

2. Translating the English to Algebra, we get:

$$4(x + 2) = 5 + 7(x - 6)$$

$$4x + 8 = 5 + 7x - 42$$

$$4x + 50 = 5 + 7x$$

$$45 = 3x$$

$$x = \mathbf{15}$$

3. If the least of the three multiples of 120 is 840, then the first multiplier must be 7, since $7 \times 120 = 840$. The next two multiples of 120 are $8 \times 120 = 960$ and $9 \times 120 = 1080$. The sum of these is $840 + 960 + 1080 = 2880$. Now, if the sum of five multiples of 96 is also 2880, then we can let n be the first multiplier and write the equation: $96n + 96(n + 1) + 96(n + 2) + 96(n + 3) + 96(n + 4) = 2880$

This can be simplified by factoring out the common 96 and then solving for n :

$$96(n + n + 1 + n + 2 + n + 3 + n + 4) = 2880$$

$$96(5n + 10) = 2880$$

$$480n + 960 = 2880$$

$$480n = 1920$$

$$n = 4$$

We want the greatest multiple of 96, which is $96(n + 4) = 96(4 + 4) = 96 \times 8 = \mathbf{768}$.

Category 6
 Team Questions
 Meet #2, November 2003

1. What is the 94th digit of the repeating decimal value of $\frac{11}{130}$?

2. Express the fraction at right as a mixed number in the form $A\frac{B}{C}$, where A , B , and C are integers and the GCF of B and C is 1.

$$9 + \frac{9}{9 + \frac{9}{9 + \frac{9}{9}}}$$

3. The number 1,000,000 has 49 positive factors. What is the sum of all the factors of 1,000,000 that contain no zero digits?

4. The LCM of two whole numbers numbers, a and b , is 283. If $a \neq b$, what is the sum $a + b$?

5. The Farey sequence of order n is the increasing sequence of all the reduced fractions with values from zero to one (inclusive) where both numerator and denominator are nonnegative integers and n is the largest denominator.

For example, the Farey sequence of order 4 is: $\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$. In the

Farey sequence of order 7, let $\frac{a}{b}$ be the 8th fraction counting from the left and

let $\frac{c}{d}$ be the 12th fraction from the left. Find the value of $\frac{a+c}{b+d}$.

Answers	
1.	_____ = A
2.	_____ = B
3.	_____ = C
4.	_____ = D
5.	_____ = E
6.	_____

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below **and then find the largest prime factor of the resulting whole number:**

$$C - 2B(A - E) - D$$

Solutions to Category 6
 Team Questions
 Meet #2, November 2003

Answers

1. 6

2. $9\frac{10}{11}$

3. 19657

4. 284

5. $\frac{1}{2}$

6. 43

1. Dividing 11 by 130 by hand, we get the decimal equivalent $0.084615\overline{3}$. After the zero, we get a six-digit repeating pattern. The 94th digit of the decimal expansion will be the 93rd digit in the repeating pattern. Since 90 is a multiple of 6, we need only find the third digit of the pattern which is **6**.

2. Starting from the bottom of the continued fraction and working our way up, we get:

$$\begin{aligned}
 9 + \frac{9}{9 + \frac{9}{9+1}} &= 9 + \frac{9}{9 + \frac{9}{10}} = 9 + \frac{9}{\frac{90}{10} + \frac{9}{10}} \\
 &= 9 + \frac{9}{\frac{99}{10}} = 9 + \frac{9}{1} \times \frac{10}{99} = 9 + \frac{10}{11} = \mathbf{9\frac{10}{11}}
 \end{aligned}$$

3. The prime factorization of 1,000,000 is $2^6 \times 5^6$. Any factor of 1,000,000 that includes a factor of 2 and a factor of 5 will certainly have a zero digit, since 2×5 equals 10. Thus, the only factors of 1,000,000 that do not have a zero digit are the six powers of 2, the six powers of 5, and 1.

The sum of the powers of 2 is:

$$2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2 + 4 + 8 + 16 + 32 + 64 = 126.$$

The sum of the powers of 5 is $5^1 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6 = 5 + 25 + 125 + 625 + 3125 + 15625 = 19530$. The final sum is $126 + 19530 + 1 = \mathbf{19657}$.

4. The only factors of the prime number 283 are 1 and 283. Since a and b are not equal and their LCM is 283, one of them must be 1 and the other 283. The desired sum is $1 + 283 = \mathbf{284}$.

5. The Farey sequence of order 7 is:

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{2}{3}, \frac{2}{5}, \frac{3}{7}, \frac{3}{4}, \frac{3}{2}, \frac{4}{3}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

If $\frac{a}{b}$ is the 8th fraction from the left, then $\frac{a}{b} = \frac{2}{5}$.

If $\frac{c}{d}$ is the 12th fraction from the left, then $\frac{c}{d} = \frac{3}{5}$.

The value of $\frac{a+c}{b+d}$ is $\frac{2+3}{5+5} = \frac{5}{10} = \frac{1}{2}$.

6. Substituting the values for A through E into the expression and evaluating, we get:

$$\begin{aligned} & C - 2B(A - E) - D \\ & 19657 - 2\left(9\frac{10}{11}\right)\left(6 - \frac{1}{2}\right) - 284 \\ & 19657 - 2\left(\frac{109}{11}\right)\left(\frac{12}{2} - \frac{1}{2}\right) - 284 \\ & 19657 - 2\left(\frac{109}{11}\right)\left(\frac{11}{2}\right) - 284 \\ & 19657 - 109 - 284 = 19264 \end{aligned}$$

Now we need to find the largest prime factor of 19264.

Dividing out all the factors of 2 first, we get:

$$\begin{aligned} 19264 \div 2 &= 9632 \\ 9632 \div 2 &= 4816 \\ 4816 \div 2 &= 2408 \\ 2408 \div 2 &= 1204 \\ 1204 \div 2 &= 602 \\ 602 \div 2 &= 301 \end{aligned}$$

Now we need to find any prime factors of 301. It's not divisible by 3 or 5, but 301 is equal to 7×43 . The prime factorization of 19264 is $2^6 \times 7 \times 43$, and **43** is the largest of these prime factors.