Meet #5
April 2003

Intermediate Mathematics League of Eastern Massachusetts

www.imlem.org

Meet #5
April 2003

You may use a calculator

Category 1
Mystery
Meet #5, April 2003

1. In his book In an Average Lifetime, author Tom Heymann claims that the average American laughs 410,078 times in an average lifetime. If the average lifespan is 74.9 years, about how many times per day does the average American laugh? Round your result to the nearest whole number.

2. If gcf(a,b) means "the greatest common factor of a and b" and $a \Diamond b$ means $ab - \text{gcf}(a,b)$, then what is the value of $30 \Diamond 24$?

3. What value of n will make the following equation true?

$$n + n^2 + \sqrt{n} = 1338$$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1
Mystery
Meet #5, April 2003

Answers

1. 15

2. 714

3. 36

1. Dividing the average number of laughs by the average lifespan, we get about 410,078 ÷ 74.9 = 5475 laughs per year. Dividing this number by the 365 days in a year, we get about 15 laughs per day. If we divide by 365.25 to include one leap day every four years, we still get about 15 laughs per day.

2. Evaluating $30 \Diamond 24$ according to the rule, we get:

$$30 \Diamond 24 = 30 \cdot 24 - \text{gcf}(30,24) = 720 - 6 = \mathbf{714}$$

3. The fact that the equation $n + n^2 + \sqrt{n} = 1338$ contains a square root and still results in a whole number answer suggests that n is a perfect square number. We should start to guess and check using some perfect square numbers such as 16, 25, 36, 49, 64, etc.

$$25 + 25^2 + \sqrt{25} = 25 + 625 + 5 = 655 \text{ (too little)}$$

$$49 + 49^2 + \sqrt{49} = 49 + 2401 + 7 = 2457 \text{ (too much)}$$

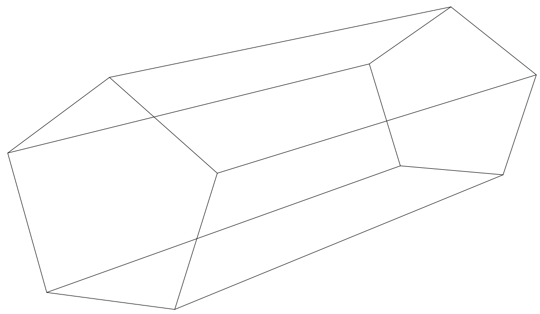
$$36 + 36^2 + \sqrt{36} = 36 + 1296 + 6 = 1338 \text{ (just right)}$$

The value of n is thus **36**.

Category 2
Geometry
Meet #5, April 2003

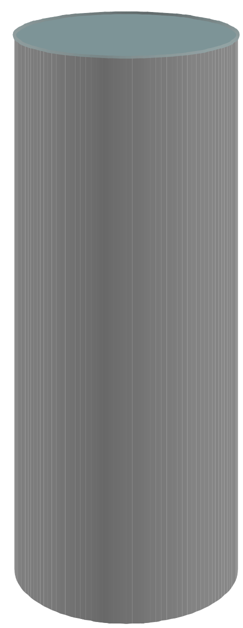
You may use a calculator

1. How many surface diagonals can be drawn on a pentagonal prism such as the one depicted below?



2. The surface area of a cube is 73.5 square inches. How many cubic inches are in its volume? Express your answer as a decimal to the nearest thousandth.

3. A cylinder that used to contain three tennis balls is now filled with water. The diameter of the cylinder is $2\frac{5}{8}$ inches and the height is $7\frac{7}{8}$ inches. A gallon is defined as a unit of liquid capacity equal to 231 cubic inches or 128 ounces. How many ounces of water does the cylinder hold? Use 3.1416 for pi and, since the cylinder can't hold more than its volume, round your answer *down* to the previous whole number of ounces.



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 2
Geometry
Meet #5, April 2003

Answers

1. 20

2. 42.875

3. 23

1. There are two surface diagonals on each of the five rectangular faces of the pentagonal prism and there are five surface diagonals on each of the two pentagonal faces of the pentagonal prism. That makes a total of $2 \times 5 + 5 \times 2 = 10 + 10 = \mathbf{20}$ surface diagonals.

2. All six sides of the cube have the same area, so the area of one side would be 73.5 ÷ 6 = 12.25 square inches. The side length of the cube is the square root of 12.25 or 3.5 inches and the volume of the cube is $3.5^3 = \mathbf{42.875}$ cubic inches to the thousandth.

3. The volume of a cylinder is given by the formula $V = \pi r^2 h$. We will need half our diameter for the radius of the cylinder, or 2.625 ÷ 2 = 1.3125 inches. Using this value for r, 3.1516 for π, and 7.875 for h, we calculate the volume of the cylinder as follows:

$$V = 3.1416 \times 1.3125^2 \times 7.875 = 42.6187 \text{ cubic inches.}$$

To convert this volume to ounces, we multiply by 128 and divide by 231. This gives us:

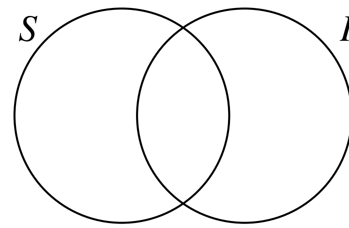
$$42.6187 \times \frac{128}{231} \approx 23.6156 \text{ ounces.}$$

Rounding this number *down* to the previous whole number of ounces, we can say that the cylinder holds about **23** ounces of water.

Category 3
Number Theory
Meet #5, April 2003

You may use a calculator

1. A set of attribute blocks includes every possible combination of three shapes (circles, squares, and triangles), three colors (red, green, and blue), and two sizes (small and large) with no duplicate blocks. Set S contains all the small blocks and set R contains all the red blocks. How many blocks are in $(S \cup R)$ but not in $(S \cap R)$?



2. Students at Yippee I. A. Middle School held three spirit days recently. Of the 40 7th graders at the school 14 participated in "Crazy Hair Day", 19 participated in "Clash Day", and 22 participated in "Funny Hat Day". Five of the students who participated in "Crazy Hair Day" also participated in "Clash Day." Ten of the students who participated in "Clash Day" also participated in "Funny Hat Day." Seven of the students who participated in "Funny Hat Day" also participated in "Clash Day". Three students participated in all three spirit days. How many 7th graders at Yippee I. A. Middle School did not participate at all in the spirit days?

3. Set A is all prime numbers.
Set B is all odd numbers.
Set C is all natural numbers that leave a remainder of 1 when divided by 6.
Set D is all natural numbers less than 40.
Set E is $((A \cap D) \cap (B \cap D)) \cup (C \cap D)$.
How many elements are there in set E?

Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 3
Number Theory
Meet #5, April 2003

Answers

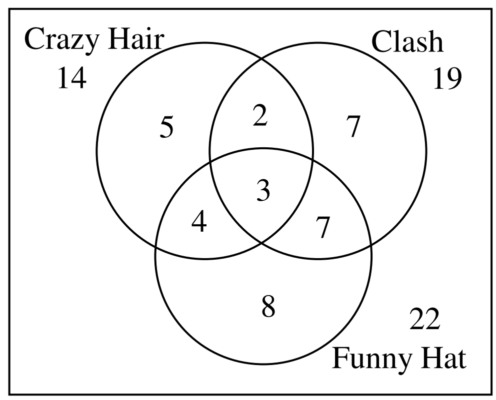
1. 9

2. 4

3. 13

1. There are $3 \times 3 \times 2 = 18$ blocks total. Set S has $3 \times 3 \times 1 = 9$ small blocks and set R has $3 \times 1 \times 2 = 6$ red blocks. There are 3 blocks in the intersection $(S \cap R)$, the small red circle, the small red square, and the small red triangle. To find the number of blocks in the union, we subtract 3 from the sum 9 + 6 to get $9 + 6 - 3 = 12$ blocks in $(S \cup R)$. This avoid double counting. If we want to exclude those three blocks entirely from our count, we must subtract 3 again to get 12 – 3 = **9** blocks in $(S \cup R)$ but not in $(S \cap R)$.

2. The Venn diagram at right can help us to keep track of the participants. The easiest number to place first is the 3 in the center which corresponds to the three students who participated in all three spirit days. The five students who participated in "Crazy Hair Day" and "Clash Day" are these three students and two others and so forth. When all the numbers are placed correctly, we see that a total of 36 students participated in the spirit days, which means that 40 – 36 = **4** did not participate at all.



3. The intersection $A \cap D$ is the prime numbers less than 40: $\{2,3,5,7,11,13,17,19,23,29,31,37\}$. The intersection $B \cap D$ is the odd numbers less than 40: $\{1,3,5,7,9,...37,39\}$. The intersection of these two sets, $(A \cap D) \cap (B \cap D)$, is just the odd prime numbers less than 40 (excluding only the number 2 from $A \cap D$ above): $\{3,5,7,11,13,17,19,23,29,31,37\}$. The intersection $C \cap D$ is all natural numbers less than 40 that leave a remainder of 1 when divided by 6: $\{1,7,13,19,25,31,37\}$. Set E is the union of the odd primes less than 40 with the natural numbers less than 40 that leave a remainder of 1 when divided by 6. This adds only the numbers 1 and 25 to the set of odd primes less than 40, giving us: $E = \{1,3,5,7,11,13,17,19,23,25,29,31,37\}$. Set E has **13** elements.

Category 4
Arithmetic
Meet #5, April 2003

You may use a calculator

1. Octahedral dice have eight congruent faces numbered 1 through 8. If two octahedral dice are rolled, what is the probability that the sum of the numbers on the top faces of the dice is at least 10 (10 or more)? Express your answer as a fraction in lowest terms.

2. Nine middle school students collected signatures to petition their principal for a field trip to the Mathematicians' Hall of Fame. The average number of signatures collected by the nine students was 23. One of the students then lost a piece of paper that had 18 signatures on it. If the average number of signatures collected by the nine students is recalculated after the loss of the paper, what would the new average be? Express your answer to the nearest whole number.

3. On the Greedy Grab game show, a contestant who answers a question correctly is blindfolded and draws two bills from a bag. If the bag was initially stocked with 1 one-hundred dollar bill, 2 fifty-dollar bills, 3 twenty-dollar bills, and 4 ten-dollar bills, what is the probability that a contestant who answers a question correctly will draw at least one hundred dollars total? Express your answer as a fraction in lowest terms.

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 4
Arithmetic
Meet #5, April 2003

Answers

1. $\frac{7}{16}$

2. 21

3. $\frac{2}{9}$

1. The table below shows the possible sums when two octahedral dice are rolled.

+	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

There are seven ways to roll a sum of 10, six ways to roll a sum of 11, five ways to roll a sum of 12, etc. Of the 8 • 8 = 64 different ways the dice can land, 28 of them produce a sum that is at least 10 (10 or more). Thus the probability of such a roll is $\frac{28}{64} = \frac{14}{32} = \mathbf{\frac{7}{16}}$.

2. The original average was 23 signatures collected per student. Since we know that nine students collected these signatures, we can compute the total number of signatures collected by multiplying 23 by 9, which is 207 signatures. Although we have no idea how many signatures were collected by any of the nine students, we know that the total went go down by 18 when one student lost the sheet of paper. The total number of signatures is now 207 – 18 = 189 and the average number of signatures per student is now 189 ÷ 9 = **21**.

3. To draw at least \$100, a contestant could draw the \$100 first, with a probability of 1/10. Or, he could draw the \$50 first (2/10), in which case the second bill must be either \$50 or \$100, with a probability of 2/10 x 2/9 = 4/90. Or, he could draw a \$10 or \$20 first (7/10), and then the \$100 on the second draw: 7/10 x 1/9 = 7/90. Adding these alternative cases gives $\frac{9}{90} + \frac{4}{90} + \frac{7}{90} = \frac{20}{90} = \mathbf{\frac{2}{9}}$

Note: Original problem was problem was ambiguous about whether the bills were replenished between drawing the bills and had an incorrect answer of 11/90. Even replenishing the bills would give 23/100. – Editor 3/2006.

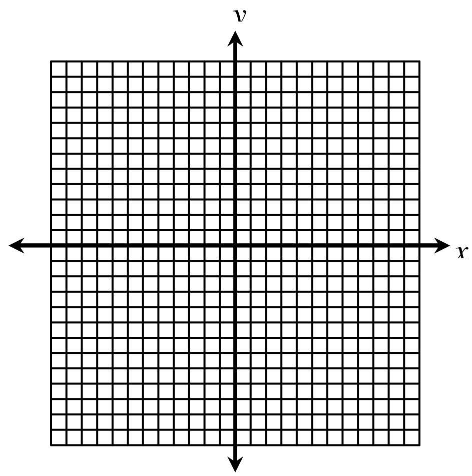
Category 5
Algebra
Meet #5, April 2003

You may use a calculator

1. The following equation has two solutions. What are they?

$$x^2 - 84 = -8x$$

2. The graph of the function $y = x^2 - 3x - 4$ crosses the x-axis at two different points. What is the average of the x-coordinates of these two x-intercepts? Express your answer as a mixed number in lowest terms.



3. The longer leg of a right triangle is nine centimeters shorter than the hypotenuse. The shorter leg is one centimeters less than half the length of the longer leg. How many centimeters are in the perimeter of the right triangle?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #5, April 2003

Answers

1. –14 and 6

2. $1\frac{1}{2}$

3. 208

1. To factor the quadratic equation $x^2 - 84x = -8x$, we first set it equal to zero as follows:

$$x^2 - 84 = -8x$$
$$+8x = +8x$$
$$x^2 + 8x - 84 = 0$$

We now seek to factor this trinomial into the product of two binomials of the form $(x \pm \quad)(x \pm \quad)$. We are looking for a pair of numbers whose product is -84 and whose *difference* is 8. (We look for a *difference* of 8 since the numbers must have the opposite sign.)

84 and 1 have a difference of 83
42 and 2 have a difference of 40
21 and 4 have a difference of 17
14 and 6 have a difference of 8
12 and 7 have a difference of 5

The pair we want is 14 and 6 and we now have the following product of two binomials equal to zero:

$$(x+14)(x-6) = 0$$

For a product to equal zero, it must be true that one or the other (or both) is equal to zero. This means we can solve the two separate equations as follows:

$$x + 14 = 0 \qquad\qquad x - 6 = 0$$
$$-14 = -14 \qquad\qquad +6 = +6$$
$$x = -14 \qquad\qquad x = 6$$

Thus the two solutions to the equation $x^2 - 84 = -8x$ are **-14** and **6**.

2. The two points at which the graph of the function $y = x^2 - 3x - 4$ cross the x-axis are known as the roots of the equation. We could determine them graphically if we create a table of well-chosen values and plot some points. It is easier to compute the roots algebraically, however, especially if we recall that the value of the y-coordinate for any point on the x-axis is zero. This means we just need to factor the equation $0 = x^2 - 3x - 4$ as we did in the previous problem.

Since the only factor pairs for 4 are 1×4 and 2×2, we should quickly find that $x^2 - 3x - 4 = (x-4)(x+1)$. As in the previous problem, this gives us a product of two binomials that is equal to zero: $(x-4)(x+1) = 0$. This happens when either $(x-4) = 0$ or when $(x+1) = 0$, so the two solutions are x = 4 and $x = -1$. These are the x-coordinates of the points where the graph of the equation crosses the x-axis. The required average of is thus $\frac{4+(-1)}{2} = \frac{3}{2} = \mathbf{1\frac{1}{2}}$.

(Note: This average also gives the x-coordinate of the vertex of the parabola and can be computed directly as $\frac{-b}{2a}$ where a and b are coefficients from the standard form of the equation $y = ax^2 + bx + c$.)

3. Let's call the hypotenuse of the right triangle h. The longer leg is h – 9 and the shorter leg is $\frac{h-9}{2} - 1$ or $\frac{h}{2} - \frac{9}{2} - 1 = \frac{h}{2} - \frac{11}{2}$. Using these three lengths in the Pythagorean Theorem, we get:

$$(h-9)^2 + \left(\frac{h}{2} - \frac{11}{2}\right)^2 = h^2$$

$$(h^2 - 18h + 81) + \left(\frac{h^2}{4} - \frac{11}{2}h + \frac{121}{4}\right) = h^2$$

To avoid working with fractions, we can multiply everything by 4. This gives us:

$$4h^2 - 72h + 324 + h^2 - 22 + 121 = 4h^2$$

Now we might subtract $4h^2$ from both sides of the equation to get:

$$-72h + 324 + h^2 - 22 + 121 = 0$$

If we combine like terms and rearrange them, this becomes:

$$h^2 - 94h + 445 = 0$$

Now we have to either factor this trinomial into the product of two binomials or use the quadratic equation. The former method is prefered when the numbers look manageable. We are looking for two numbers whose product is 445 and whose sum is 94. It turns out that $445 = 5 \times 89$ and in fact 5 + 89 = 445. In other words, $h^2 - 94h + 445 = (h-5)(h-89) = 0$. The two solutions are thus h = 5 and h = 89. Only the second solution makes sense for the hypotenuse of our right triangle, since the first solution would create a negative lengths for the longer leg. Now that we know h = 89 centimeters, we can determine that the longer leg is 89 – 9 = 80 centimeters and the the shorter leg is 80 ÷ 2 – 1 = 39 centimeters. The perimeter of the triangle is thus 89 + 80 + 39 = **208** centimeters.

Editor's note: The original problem had inches mixed in by mistake.

Category 6
Team Questions
Meet #5, April 2003

You may use a calculator

1. A palindrome is a number that reads the same backward and forward. What is the largest prime palindrome less than 1000?

2. It takes Cliff four hours to stack a chord of wood by himself. It takes Phil three hours to stack a chord of wood by himself. How long will it take them to stack a chord of wood if they work together, each working at his individual rate? Express your answer in minutes to the nearest whole number.

3. The areas of three faces of a rectangular prism are 45 square inches, 50 square inches, and 90 square inches. If all side lengths are whole numbers of inches, how many cubic inches are in the volume of the rectangular prism?

4. A 3-digit number between 200 and 300 (exclusive) is chosen at random. What is the probability that the number is divisible by one of its digits? Express your result as a reduced fraction.

5. The roots of the equation $x^2 - 5x = b$ are 14 and a. Find the value of b.

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. __________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\left(\frac{3(A+17)}{\sqrt{A-B+2C+E-3}}\right)\times D$$

Solutions to Category 6
Team Questions
Meet #5, April 2003

Answers

1. 929

2. 103

3. 450

4. $\frac{26}{33}$

5. 126

6. 52

1. If there is a prime palindrome in the nine hundreds, then it will begin with a 9 and end with a 9. We can quickly eliminate 999, 969, 939, and 909, since they are divisible by 3. That leaves 989, 979, 959, 949, 929, and 919 in the nine hundreds. None of these numbers is divisible by 2, 3, or 5. Also, we will only need to test for divisibility by primes up to 31, since 32 squared is 1024, which is larger than all our candidates. That means we must try dividing each of them by 7, 11, 13, 17, 19, 23, 29 and 31. (Good thing this is a team question.) The composites among them would be found as follows: 989 ÷ 23 = 43, 979 ÷ 11 = 89, 959 ÷ 7 = 137, and 949 ÷ 13 = 73. The next number on our list, **929**, is not divisible by any of our test numbers, therefore it must be the largest prime palindrome less than 1000.

2. In one hour, Cliff would stack one third of a chord and Phil would stack one fourth of a chord. Together they would stack $\frac{1}{3}+\frac{1}{4}=\frac{3}{12}+\frac{4}{12}=\frac{7}{12}$ of a chord in one hour. This is a little more than half a chord, so it should take them just under two hours. In fact, it would take them $\frac{12}{7}$ of an hour to stack a full chord at this rate, since $\frac{7}{12}\times\frac{12}{7}=\frac{84}{84}=1$. Multiplying $\frac{12}{7}$ by the 60 minutes there are in an hour, we get $\frac{12}{7}\times 60=\frac{720}{7}=102\frac{6}{7}$, or nearly **103** minutes.

3. Since all the side length of the rectangular prism are whole numbers, we will have to work with the paired factors of 45, 50, and 90, shown at right. The factor of 10 is common to 50 and 90; 9 is common to 45 and 90; and 5 is common to 45 and 50. The dimensions of the rectangular prism are thus 10, 9, and 5, for a volume of $10\times 9\times 5=$**450** cubic inches.

45×1
15×3
9×5

50×1
25×2
10×5

90×1
45×2
30×3
18×5
15×6
10×9

4. There are 99 3-digit numbers between 200 and 300. All 49 of the even numbers are divisible by the 2 in the hundreds place. All 10 of the numbers ending in 1 are divisible by that 1 and all 10 of the numbers ending in 5 are divisible by the 5. So far, at least 69 of the 99 3-digit numbers are divisible by one of their digits. The only numbers that require further checking are the 30 numbers that end in 3, 7, or 9.

We can eliminate 213, 217, and 219 since they are each divisible by the 1 in the tens place. For other numbers ending in 3, if the sum of the digits is a multiple of 3, the number is divisible by the 3. Similarly, for numbers ending in 9, if the sum of the digits is a multiple of 9, the number is divisible by the 9. For the numbers ending in 7, we can subtract an easy multiple of 7 to see if the remainder is a multiple of 7, as with 287 – 210 = 77.

In the end, we should find that 9 of the 30 numbers ending in 3, 7, or 9 are divisible by one of their digits. These are: 213, 217, 219, 237, 243, 273, 279, 287, and 297. That makes a total of 69 + 9 = 78 out of 99 numbers that are divisible by one of their digits, for a probability of $\frac{78}{99}$, which reduces to $\mathbf{\frac{26}{33}}$.

(The twenty-one numbers that are not divisible by one of their digits are: 203, 207, 209, 223, 227, 229, 233, 239, 247, 249, 253, 257, 259, 263, 267, 269, 277, 283, 289, 293, and 299.)

5. To solve a quadratic equation we first must set it equal to zero and then either factor it into the product of two binomials or use the quadratic equation. In this case, the equation $x^2 - 5x = b$ is easily converted to $x^2 - 5x - b = 0$, but in place of a known constant value, we have the unknown value *b*. We do know that the roots are 14 and *a*, however, which tells us that our trinomial would factor into the product of two binomials as follows:

$x^2 - 5x - b = (x - 14)(x - a)$.

If we were to multiplying the two binomials on the right using the F.O.I.L. method, we would get:

$$x^2 - 5x - b = x \cdot x + x \cdot (-a) + (-14) \cdot x + (-a)(-14).$$

The trick here is to notice that the so-called Outer and Inner products on the right combine to make the middle term in the trinomial on the left. In other words, $-ax - 14x = -5x$, which can be solved for *a*. If we add 14*x* to both sides of this equation, we find that $-ax = 9x$, so *a* must equal –9. We can now go back to the F.O.I.L. equation above, where the Last product, $(-a)(-14)$ must equal –*b*. Evaluating this for *a* = –9, we get $(-a)(-14) = (-(-9))(-14) = -126$. (Be careful with all the negative signs!) If –126 = –*b*, then *b* must equal **126**.

6. Substituting the values for *A* through *E* into the expression, we get:

$$\left(\frac{3(A+17)}{\sqrt{A-B+2C+E-3}}\right)\times D=\left(\frac{3(929+17)}{\sqrt{929-103+2\times450+126-3}}\right)\times\frac{26}{33}$$

$$=\left(\frac{3(946)}{\sqrt{929-103+900+126-3}}\right)\times\frac{26}{33}$$

$$=\left(\frac{3(946)}{\sqrt{1849}}\right)\times\frac{26}{33}$$

$$=\frac{3(946)}{43}\times\frac{26}{33}$$

$$=\frac{(946)}{43}\times\frac{26}{11}$$

$$=\frac{22}{1}\times\frac{26}{11}$$

$$=2\times26$$

$$=\mathbf{52}.$$