Meet # 1
October, 2002

Intermediate Mathematics League of Eastern Massachusetts

www.imlem.org

Meet # 1
October, 2002

Category 1
Mystery
Meet #1, October, 2002

1. Oscar lined up 11 number cards as shown below.

$\boxed{0}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}$

He then moved three cards, one at a time, from the left end of the line to the right end of the line. Next, his friend Phelix moved some more cards, one at a time, from the left end of the line to the right end of the line. After Phelix moved his cards, the third card from the right end of the line was the 5 card. How many cards did Phelix move?

2. What is the maximum possible value for $a^b \times c$, if *a*, *b*, and *c* are replaced by the numbers 2, 3, and 5?

3. Use the clues to find out the ZIP code.

___	___	___	___	___	–	___	___	___	___
a	*b*	*c*	*d*	*e*		*f*	*g*	*h*	*i*

Each of the digits is used only once.

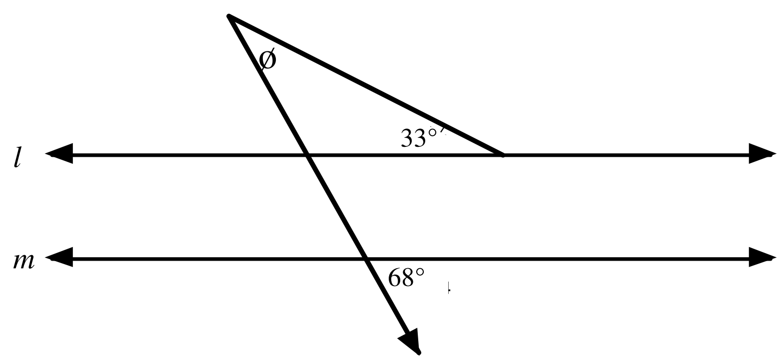
b, *d*, *f*, and *h* are prime.
a, *c*, *d*, and *e* are even.
e, g, and *i* are perfect squares
c, d, e, and *g* are powers of two.
$a + b = b$
$4b - h = f$

Answers
1. ________________
2. ________________
3. ________________

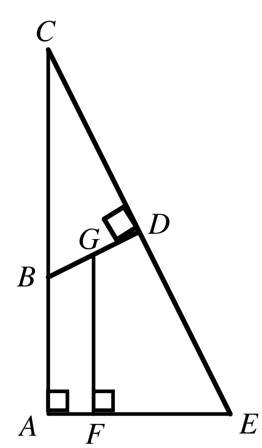
Category 2
Geometry
Meet #1, October, 2002

1. The measure of a certain angle *a* is 39 degrees. Let *s* be the supplement of this angle *a* and let *c* be the complement of angle *a*. How many degrees are there in the measure of angle (*s* + *c*)?

2. Line *l* is parallel to line *m*. Find the measure of angle ø in degrees.



3. In the figure shown at right, angles *CAE, GFE,* and *CDB* are right angles and angle *ACE* measures 27 degrees. How many degrees are in the measure of the angle *DGF*?



Answers
1. ________________
2. ________________
3. ________________

Category 3
Number Theory
Meet #1, October, 2002

1. If the five-digit number 837A5 is known to be divisible by 15, what is the sum of all the possible values of the digit A?

2. Twin primes are two primes that have a positive difference of 2, such as 11 and 13. What is the sum of the least twin primes greater than 50?

3. Find the value of n, if
n is a natural number,
n is less than 100,
n is the product of two primes,
the sum of the digits of n is 10, and
the positive difference between the two prime factors of n is a multiple of 13.

Answers
1. _______________
2. _______________
3. _______________

Category 4
Arithmetic
Meet #1, October, 2002

1. Evaluate the expression below. Express your result as a decimal rounded to the nearest tenth.

$$\frac{38-3\times 2^2+5(43-38)}{72\div 4\times 2}$$

2. The line plot below shows the ages of the students who went on the ski trip. Each X represents a person with the age indicated by the number below it. If *A* is the mode of the data, *B* is the median of the data, and *C* is the number of students on the trip, find the value of $\frac{B^2-A^2}{C}$. Express your answer as a mixed number in simplest form.

			X				
			X	X			
			X	X	X		
		X	X	X	X	X	X
X	X	X	X	X	X	X	X
8	9	10	11	12	13	14	15

Age in Years of Students on Ski Trip

3. Max's average on his first six quizzes was 88. After two more quizzes, his average was 90. What is the average of his 7th and 8th quiz scores?

Answers

1. _______________
2. _______________
3. _______________

Category 5
Algebra
Meet #1, October, 2002

1. Simplify the expression below.

$$5x+4(3x+5)+7(3x+5)-9(3x+5)-4x-13$$

2. Evaluate the expression below for $x=\frac{3}{4}$ and $y=\frac{5}{6}$.

$$12(2x+y)-3(2x-y)$$

3. Find the value of N that will make the equation below an identity.
(An identity is an equation for which all real numbers are solutions.)

$$9x-3(x+2)=2(3x+8)-2N$$

Answers
1. ________________
2. ________________
3. N = ____________

Category 6
Team Questions
Meet #1, October, 2002

1. How many sets of twin primes are there between 10 and 100?

2. What is the least composite number with exactly 8 factors?

3. A set of seven natural numbers has a mode of 28, a median of 25, and a mean of 26. What is the greatest number that could be one of the numbers in the set?

4. A series of regular congruent pentagons are placed edge to edge to form a ring with a polygon interior. How many sides are on the regular polygon that is the interior of this ring of pentagons?

5. Every cubic number can be written as the sum of a set of consecutive odd numbers. What is the least number in the set of consecutive odd numbers that has a sum that is equal to 6^3?

Answers	
1. ______________	$= A$
2. ______________	$= B$
3. ______________	$= C$
4. ______________	$= D$
5. ______________	$= E$
6. ________________	

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\sqrt{A+2B+2C+2D+2E-2}$$

Solutions to Category 1
Mystery
Meet #1, October, 2002

Answers

1. 5

2. 486

3. 03824-7159
or 03824-5179

1. The line of cards starts like this:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

After Oscar has moved three cards, the line is:

3, 4, 5, 6, 7, 8, 9, 10, 0, 1, 2

If the third card from the right is to be the 5, Phelix must move five (**5**) cards and the line looks like this:

8, 9, 10, 0, 1, 2, 3, 4, 5, 6, 7

2. The maximum possible value for $a^b \times c$, if *a*, *b*, and *c* are replaced by the numbers 2, 3, and 5, is **486**.

3. Using the second clue, we know that the candidates for *b*, *d*, *f*, and *h* must be the single digit primes 2, 3, 5, and 7. Similarly, *a*, *c*, *d*, and *e* must be the evens 0, 2, 4, 6, and 8. Since 2 is the only even prime, we know that *d* must be 2. The candidates for *e, g,* and *i* are the perfect squares 1, 4, and 9. Since 4 is the only even single-digit square, we know that *e* must be 4. The candidates for *c, d, e,* and *g* are the powers of two 1, 2, 4, and 8. The digits 4 and 2 are already assigned, so *c* must be the even number 8, which makes *g* equal 1. That leaves *i* equal to 9 as the only remaining square. The equation $a + b = b$, tells us that *a* can only be zero. The remaining letters *b*, *f*, and *h* must be the primes 3, 5, and 7. Trying these three values in the equation that is the final clue, we find that $4b - h = f$ is true when *b* = 3*, f* = 7, and *h* = 5. The ZIP code is thus **03824-7159**.

Editor's note:
03824-5179 also works because (12-7=5 and 12-5=7)

Solutions to Category 2
Geometry
Meet #1, October, 2002

Answers

1. 192

2. 35

3. 117

1. If s is the supplement of a 39 degree angle, then $s = 180 - 39 = 141$. If c is the complement of angle a, then $c = 90 - 39 = 51$. Thus the value of $s + c$ is 141 + 51 = **192**.

2. The obtuse angle in the triangle is the supplement of 68 degrees, or 180 – 68 = 112 degrees. Since every triangle has a total of 180 degrees, ø must be 180 – 112 – 33 = **35** degrees. A shorter way to arrive at this result is 68 – 33 = 35, since an exterior angle of a triangle is equal to the sum of the two non-adjacent angles.

3. If the measure of angle *ACE* is 27 degrees, then the measure of angle *AEC* must be 90 – 27 = 63 degrees. The angle sum in quadrilateral *DEFG* must be 360 degrees. Angles *EFG* and *EDG* are right angles and *AEC* is 63 degrees. Thus the measure of angle *DGF* is 360 – 90 – 90 – 63 = **117** degrees.

Solutions to Category 3
Number Theory
Meet #1, October, 2002

Answers

1. 12

2. 144 <u>or 120</u>

3. 82

1. To be sure a number is divisible by 15, it must be divisible by both 3 and 5. The five-digit number 837*A*5 is definitely divisible by 5, since its units digit is a 5. If it is to be divisible by 3, the sum of its digits must be a multiple of 3. The known digits have a sum of 8 + 3 + 7 + 5 = 23. This means that *A* can have the values 1, 4, or 7. The sum of these possible values is 1 + 4 + 7 = **12**.

2. There are no twin primes in the fifties or the sixties. Thus the least twin primes greater than 50 are 71 and 73. Their sum is **144**.
Editor note: The original meet had 144 as the answer, but 59 and 61 are also twin primes, so the correct answer should be 120. Thanks to Zhiping You for pointing this out.

3. There are too many possible products of two primes to start with that clue. The natural numbers less than 100 with a sum of digits equal to 10 are 19, 28, 37, 46, 55, 64, 73, 82, and 91. Of these, four of these are the product of two primes: 46 = 2 • 23, 55 = 5 • 11, 82 = 2 • 41, and 91 = 7 • 13. Since the difference between the prime factors of 82 is 41 – 2 = 39 and 39 is a multiple of 13, *n* must be **82**.

Solutions to Category 4
Arithmetic
Meet #1, October, 2002

Answers

1. 1.4

2. $1\frac{3}{20}$

3. 96

1. Following the order of operations to evaluate the expression, we get:

$$\frac{38-3\times 2^2+5(43-38)}{72\div 4\times 2}=\frac{38-12+5(5)}{18\times 2}=\frac{26+25}{36}$$

$\frac{51}{36}=\frac{17}{12}=$ about **1.4** to the nearest tenth.

2. The line plot shows a mode of 11, a median of 12, and the number of students is 20. Thus, the expression $\frac{B^2-A^2}{C}$ can be evaluated as follows:

$$\frac{12^2-11^2}{20}=\frac{144-121}{20}=\frac{23}{20}=\mathbf{1\frac{3}{20}}$$

3. If Max's average on his first six quizzes was 88, we know that the sum of the six quiz scores was $88\times 6=528$. Similarly, after eight quizzes, when his average was 90, the sum of his 8 quiz scores must have been $90\times 8=720$. The difference, $720-528=192$, is the sum of the seventh and eighth quiz scores. The average of these two is thus 192 ÷ 2 = **96**.

Solutions to Category 5
Algebra
Meet #1, October, 2002

Answers

1. $7x - 3$

2. 26

3. $N = 11$

1. To simplify the expression, we could use the distributive property on each set of parentheses as follows:

$$5x + 4(3x+5) + 7(3x+5) - 9(3x+5) - 4x - 13$$
$$= 5x + 12x + 20 + 21x + 35 - 27x - 45 - 4x - 13$$

Then we combine like terms and get the simplified expression **$7x - 3$**. Alternatively, we might notice that all three sets of parentheses contain the same $3x + 5$. Thus there are $4 + 7 - 9 = 2$ of these, so the original expression can be rewritten as: $5x + 2(3x+5) - 4x - 13$.

This simplifies to: $5x + 6x + 10 - 4x - 13 = \mathbf{7x - 3}$.

2. Replacing the *x's* and the *y*'s in the expression with the values given, we get:

$$12\left(2\cdot\frac{3}{4}+\frac{5}{6}\right) - 3\left(2\cdot\frac{3}{4}-\frac{5}{6}\right) = 12\left(2\cdot\frac{9}{12}+\frac{10}{12}\right) - 3\left(2\cdot\frac{9}{12}-\frac{10}{12}\right)$$
$$= 12\left(\frac{18}{12}+\frac{10}{12}\right) - 3\left(\frac{18}{12}-\frac{10}{12}\right) = 12\left(\frac{28}{12}\right) - 3\left(\frac{8}{12}\right) = 28 - 2 = \mathbf{26.}$$

3. First, we can distribute and combine like terms: $9x - 3(x+2) = 2(3x+8) - 2N$

$$9x - 3x - 6 = 6x + 16 - 2N$$
$$6x - 6 = 6x + 16 - 2N$$

With $6x$ on each side of the equation, we are guaranteed the same amount on each side for all real values of x. We only need to find the value of N that will make everything else equal so the equation will be an identity.

$$-6 = 16 - 2N$$
$$2N - 6 = 16$$
$$2N = 22$$
$$\mathbf{N = 11}$$

Solutions to Category 6
Team Questions
Meet #1, October, 2002

Answers

1. 6

2. 24

3. 95

4. 10

5. 31

6. 18

1. The six (**6**) sets of twin primes between 10 and 100 are: 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, and 71 and 73.

2. Numbers with exactly 8 factors have a prime factorization that is either the seventh power of a prime or the product of a prime and the cube of a different prime. Using the smallest primes, we get $2^7 = 128$ for the first type and $2^3 \cdot 3 = 8 \cdot 3 = 24$ for the second type. The eight factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

3. Since the mode of the number is 28, at least two of the seven numbers must be 28. We can be sure that one of the numbers is 25 since that is the median and we have an odd number of elements in the set. We know three of the seven numbers. If the mean of the seven numbers is 26, then the sum of the numbers is 7 • 26 = 182. To find the greatest possible number that could be in the set, lets make the least possible numbers for the other three unknowns. They must be natural numbers, but there cannot be more than one of each or else the mode of the set would no longer be 28. The smallest six numbers could be 1, 2, 3, 25, 28, and 28, which have a sum of 87. The greatest number could be 182 – 87 = **95**.

4. Every vertex of a regular pentagon has an interior angle measure of 108 degrees. When two pentagons come together edge to edge, there are 108 + 108 = 216 interior degrees at each vertex. This leaves 360 – 216 = 144 degrees in the interior of the ring at each vertex. The task now is to find the regular polygon that has an interior angle of 144 degrees. If the interior angle is 144, then the exterior angle is 180 – 144 = 36 degrees. Since 360 ÷ 10 = 36, a 10-sided polygon, or a decagon, has and exterior angle of 36 and thus an interior angle of 144 degrees. There are ten (**10**) sides on the regular polygon that is the interior of the ring of pentagons.

5. This question is more easily answered if we start with a simpler question and look for a pattern. The first six cubic numbers are 1, 8, 27, 64, 125, and 216. The number 1 can written as the sum of one odd number, namely 1. Next, 8, which is 2 cubed, can be written the sum of two consecutive odd number as 3 + 5. 27 is the sum of three consecutive odd numbers, 7 + 9 + 11. 64 is the sum of four consecutive odd numbers, 13 + 15 + 17 + 19. 125 is the sum of five consecutive odd numbers, 21 + 23 + 25 + 27 + 29. And finally, 216 is the sum of six consecutive odd numbers, 31 + 33 + 35 + 37 + 39 + 41. The least number in this set is **31**.

6. Substituting the values for *A* through *E* into the expression $\sqrt{A+2B+2C+2D+2E-2}$, we get:

$$\sqrt{6+2\times 24+2\times 95+2\times 10+2\times 31-2}$$

$$=\sqrt{6+48+190+20+62-2}$$

$$=\sqrt{324}= \mathbf{18.}$$