Intermediate Mathematics League of Eastern Massachusetts

Meet #2
November, 2001

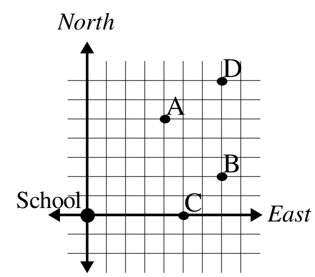
Category 1
Mystery
Meet #2, November, 2001

1. A soldier was standing at attention facing North when he was given the following sequence of commands in quick succession:

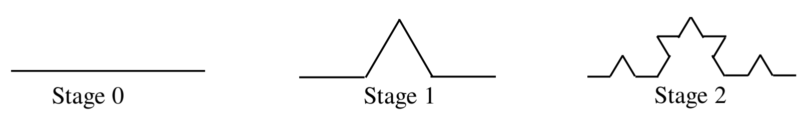
Left face! Left face! Left face! Right face! About face!
Left face! About face! Left face! Left face!

If "left face" means turn 90 degrees to the left, "right face" means turn 90 degrees to the right, and "about face" means turn 180 degrees around, which direction was the soldier facing after executing all these commands?

2. Jane, Tara, Peter, and Marina live in different houses and go to the same school. Use the clues and the graph to find out which point represents Marina's house.
a. Tara and Peter both live seven blocks east of the school.
b. Peter and Jane walk the same number of blocks to school.
c. Marina likes strawberry ice cream.



3. If the total length of the line is 1 inch at stage 0 and $1\frac{1}{3}$ inches at stage 1, how many inches long will the line be at stage 4? Express your answer as a mixed number in lowest terms.

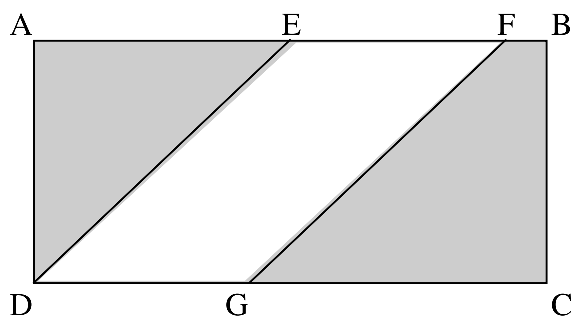


Answers
1. _______________
2. _______________
3. _______________

Category 2
Geometry
Meet #2, November, 2001

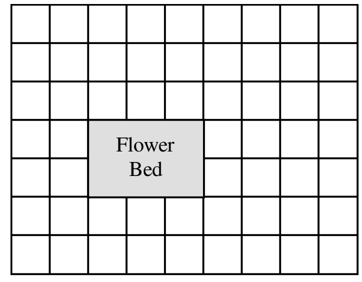
1. A rectangle is one foot longer than it is wide and has a perimeter of 50 feet. How many square feet are in the area of the rectangle?

2. In the rectangle below, the measure of $\overline{AB}$ is 24 inches, the measure of $\overline{AD}$ is 11 inches, and the measure of $\overline{EF}$ is 10 inches. If $\overline{ED}$ is parallel to $\overline{FG}$, how many square inches are in the combined area of the shaded regions.



3. Larry has created a backyard patio using bluestone tiles that measure 2 feet by 2 feet. He laid out the tiles as shown in the diagram below. Notice that he has left room for a flower bed in the middle of the patio. He now plans to run plastic edging around both the outer and inner perimeters of the tiled area. How many linear feet of plastic edging will be required?

Answers
1. _______________
2. _______________
3. _______________



Category 3
Number Theory
Meet #2, November, 2001

1. What is the smallest positive composite number that does not have any prime factors less than 20?

2. The greatest common factor of x and y is 5 and the least common multiple of x and y is 525. If $x = 75$, what is the value of y?

3. Of all the whole numbers between 60 and 70 (not including 60 and 70) which has the greatest number of factors?

Answers

1. _______________
2. _______________
3. _______________

Category 4
Arithmetic
Meet #2, November, 2001

1. What fraction is $33\frac{1}{3}\%$ greater than $\frac{3}{5}$?
Express your answer in lowest terms.

2. Simplify the expression: $\frac{0.\overline{5}}{0.8\overline{3}}$. Write your answer as a fraction in lowest terms.

3. How many simple fractions of the form $\frac{n}{15}$, where n is a natural number and $0 < n \leq 15$, have a decimal equivalent that eventually goes to repeating 3's?

Answers
1. _______________
2. _______________
3. _______________

Category 5
Algebra
Meet #2, November, 2001

1. Five consecutive multiples of 3 have a sum of 465. What is the product of the second and the fourth of these five numbers?

2. Three years from now, Martha will be two and a half times as old as she was 6 years ago. How old is she now?

3. The formula $n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\ldots$, where *n* is a natural number and p_1, p_2, p_3, etc. are the different prime factors of *n*, can be used to determine how many of the positive whole numbers less than *n* are relatively prime to *n*. If *a* is the number of positive whole numbers less than 600 that are relatively prime to 600 and *b* is the number of positive whole numbers less than 900 that are relatively prime to 900, then find $b-a$.

Answers
1. ______________
2. ______________
3. ______________

Category 6
Team Questions
Meet #2, November, 2001

1. If A is the number of primes less than 20 and B is the number of primes between 20 and 50, find $A - B$.

2. Together Tom and Ralph weigh 347 pounds. Ralph and Sam weigh 395 pounds together. Sam and Jorge weigh 339 pounds together. How many pounds do Jorge and Tom weigh together?

3. If you add $\frac{1}{8}$ of the price of an apple to the price of the apple, you get an amount equal to three quarters of the change you would get if you bought the apple with a dollar (no tax). What is the cost of the apple in cents?

4. What is the remainder when 3^{459} is divided by 10?

5. Each person in a group of tourists bought the same postcard at the lighthouse gift shop. If each person paid exact change using 4 coins and the total value of the postcards was $3.77, how many pennies did the gift shop collect?

Answers

1. __________ = A
2. __________ = B
3. __________ = C
4. __________ = D
5. __________ = E
6. ____________

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$C - \frac{3E + \sqrt{B - 2A} + 2A}{D}$$

Solutions to Category 1
Mystery
Meet #2, November, 2001

Answers

1. west

2. C

3. $3\frac{13}{81}$

1. The soldier faces west, then south, then east, then south, then north, then west, then east, then north, and finally **west**.

2. From clue a, we know that points D and B represent Tara and and Peter's houses. From clue b, we know that points A and B represent Peter and Jane's houses. Combining these clues, we can determine that Peter must live at point B. This means Tara must live at D and Jane must live at A. Marina must live at point C regardless of ice cream preference.

3. The length of the line is growing at each stage by a factor of $\frac{4}{3}$. At stage 2, the line will be $\left(\frac{4}{3}\right)^2 = \frac{16}{9} = 1\frac{7}{9}$ inches long. At stage 3, it will be $\left(\frac{4}{3}\right)^3 = \frac{64}{27} = 2\frac{10}{27}$ inches long. And at stage 4, it will be $\left(\frac{4}{3}\right)^4 = \frac{256}{81} = \mathbf{3\frac{13}{81}}$ inches long.

Solutions to Category 2
Geometry
Meet #2, November, 2001

Answers

1. 156

2. 154

3. 84

1. If the rectangle were not one foot longer than it is wide, it would be a square with a perimeter of 48 feet and therefore a side length of 12 feet. The dimensions of the rectangle must be 12 feet by 13 feet, which give it an area of **156** square feet.

2. It is not necessary to know where points E and F fall on $\overline{AB}$. Rectangle ABCD has an area of $24 \times 11 = 264$ square inches and parallelogram DEFG has an area of $10 \times 11 = 110$ square inches. The area of the shaded regions is what remains when parallelogram DEFG is removed from rectangle ABCD: $264 - 110 = \mathbf{154}$ square inches.

3. We see outer dimensions of 7 tiles by 9 tiles, which means 14 feet by 18 feet since the tiles measure 2 feet by 2 feet. The total outer perimeter is thus $2(14+18) = 2 \times 32 = 64$ feet. The inner perimeter (around the flower bed) measures 2 tiles by 3 tiles, or 4 feet by 6 feet for an inner perimeter of $2(4+6) = 2 \times 10 = 20$ feet. Larry will need $64 + 20 = \mathbf{84}$ feet of plastic edging.

Solutions to Category 3
Number Theory
Meet #2, November, 2001

Answers

1. 529

2. 35

3. 66

1. The smallest positive composite number that does not have any prime factors less than 20 must be the square of the first prime greater than 20, or 23^2 which is **529**.

2. The prime factorization of 525 is $3 \times 5^2 \times 7$ and the prime factorization of 75 is 3×5^2. Whatever *y* is, it must have the common factor of 5 and it must also have all the factors of the LCM that aren't accounted for by *x*. Thus *y* must equal 5×7 or **35**. In general, the product of the LCM and the GCF of two numbers is equal to the product of the two numbers.

3. Consider the prime factorizations of the whole numbers 61 through 69. The number of factors can be calculated by raising each exponent by one and finding their product.

$61 = 61^1$ and has just 2 factors.
$62 = 2^1 \times 31^1$ and has $2 \times 2 = 4$ factors.
$63 = 3^2 \times 7^1$ and has $3 \times 2 = 6$ factors.
$64 = 2^6$ and has 7 factors.
$65 = 5^1 \times 13^1$ and has $2 \times 2 = 4$ factors.
$66 = 2 \times 3 \times 11$ and has $2 \times 2 \times 2 = 8$ factors.
$67 = 67^1$ and has just 2 factors.
$68 = 2^2 \times 17^1$ and has $3 \times 2 = 6$ factors.
$69 = 3^1 \times 23^1$ and has $2 \times 2 = 4$ factors.

66 has the greatest number of factors.

Solutions to Category 4
Arithmetic
Meet #2, November, 2001

Answers

1. $\frac{4}{5}$

2. $\frac{2}{3}$

3. 5

1. Some students may notice that $33\frac{1}{3}\%$ of $\frac{3}{5}$ has got to be $\frac{1}{5}$ since it's one of three equal parts. Thus $\mathbf{\frac{4}{5}}$ is the desired fraction that is $33\frac{1}{3}\%$ greater than $\frac{3}{5}$. Otherwise, one could multiply the fraction $\frac{3}{5}$ by the fraction equivalent of $133\frac{1}{3}\%$, which is $\frac{4}{3}$. Thus, we get $\frac{4}{3}\times\frac{3}{5}=\frac{12}{15}=\frac{4}{5}$.

2. Repeating decimals can be converted to fractions as follows: Let $x = 0.\overline{5}$ and $10x = 5.\overline{5}$. Then we have $10x - x = 9x$ and $5.\overline{5} - 0.\overline{5} = 5$. This means that $9x = 5$, or $x = \frac{5}{9}$. Similarly, let $y = 0.8\overline{3}$ and $10y = 8.3\overline{3}$. Then we have $10y - y = 9y$ and $8.3\overline{3} - 0.8\overline{3} = 7.5$. Thus $9y = 7.5$, or $y = \frac{7.5}{9} = \frac{75}{90} = \frac{5}{6}$. We can now rewrite our original fraction $\frac{0.\overline{5}}{0.8\overline{3}}$ as $\frac{\frac{5}{9}}{\frac{5}{6}}$ and simplify as follows: $\frac{5}{9}\div\frac{5}{6}=\frac{5}{9}\times\frac{6}{5}=\frac{30}{45}=\mathbf{\frac{2}{3}}$.

3. In converting fifteenths to decimal equivalents, we find that the factor of 5 poses no problem in our base ten decimal system. It is the factor of 3 that causes repeating digits. Of the fifteen values of *n* to be used as numerators, five of them will result in terminating decimals, 5 will result in repeating 6's, and five (**5**) will result in repeating 3's. The five with repeating 3's are:

$\frac{2}{15}=0.1\overline{3}$, $\frac{5}{15}=0.3\overline{3}$, $\frac{8}{15}=0.5\overline{3}$, $\frac{11}{15}=0.7\overline{3}$, $\frac{14}{15}=0.9\overline{3}$

Solutions to Category 5
Algebra
Meet #2, November, 2001

Answers

1. 8640

2. 12

3. 80

1. Dividing 465 by 5, we find that the middle of these five consecutive multiples of 3 must be 93. The numbers are 87, 90, 93, 96, and 99. The product of the second and the fourth of these numbers is $90 \times 96 = \mathbf{8640}$.

2. Let x be Martha's current age. Converting the words to algebra, we get:

$$x + 3 = 2.5(x - 6)$$
$$x + 3 = 2.5x - 15$$
$$x + 18 = 2.5x$$
$$18 = 1.5x$$
$$x = \tfrac{18}{1.5}$$
$$x = \mathbf{12}$$

3. To use the formula, we must first find the prime factorizations for both 600 and 900. $600 = 2^3 \cdot 3 \cdot 5^2$ and $900 = 2^2 \cdot 3^2 \cdot 5^2$. To calculate a (the number positive whole numbers less than 600 that are relatively prime to 600), we get $a = 600\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{5}\right) = 600 \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{4}{5}$ $= 600 \cdot \tfrac{4}{15} = 40 \cdot 4 = 160$. Similarly, we get $b = 900\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{5}\right) = 900 \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{4}{5}$ $= 900 \cdot \tfrac{4}{15} = 60 \cdot 4 = 240$. Finally, we can calculate $b - a = 240 - 160 = \mathbf{80}$.

Solutions to Category 6
Algebra
Meet #2, November, 2001

Answers

1. 1

2. 291

3. 40

4. 7

5. 87

6. 0

1. The primes less than 20 are: 2, 3, 5, 7, 11, 13, 17, and 19, which tells us that $A = 8$. The primes between 20 and 50 are: 23, 29, 31, 37, 41, 43, and 47, so $B = 7$. Thus $A - B =$ **1**.

2. If Tom and Ralph weigh 347 pounds and Sam and Jorge weigh 339 pounds, then all four men together must weigh $347 + 339 = 686$ pounds. If we take away the combined weight of Ralph and Sam (395 pounds), we're left with $686 - 395 = \mathbf{291}$ pounds, which must be the combined weight of Jorge and Tom.

3. Let x equal the price of the apple in cents. Converting the words to algebra, we get

$$\tfrac{1}{8}x + x = \tfrac{3}{4}(100 - x).$$

Simplifying this, we get:

$$\tfrac{9}{8}x = 75 - \tfrac{3}{4}x$$

$$\tfrac{15}{8}x = 75$$

$$x = 75 \cdot \tfrac{8}{15} = 5 \cdot 8 = 40.$$

The apple costs **40** cents.

4. When a number is divided by 10, the remainder is the digit in the ones place. The powers of 3 generate a repeating pattern of four different digits in the ones place. $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$, etc. The digit in the ones place goes 3, 9, 7, 1, 3, 9, 7, 1, etc. Thus it is not necessary to compute the value of 3^{459} to find out which of these four digits is in the ones place. We only need to find the remainder when 459 it is divided by 4 to determine which of the four possible digits ends up in the ones place of 3^{459}. We find that $459 = 4 \times 114 + 3$, so the remainder will be 3. This means 3^{459} will end in the third number of our cycle which is 7. And since 7 is the digit in the ones place, **7** is the remainder when the number is divided by 10.

5. Since a number of people paid the exact same amount, we must find some factors of 377 ($3.77 = 377 cents). We find that 377 is the product of two primes, namely 29 and 13. It is not possible to make 29 cents with just four coins, so it must be that 29 people bought postcards for 13 cents, each person using a dime and three pennies. Thus the gift shop collected $29 \times 3 = \mathbf{87}$ pennies.

6. Substituting the correct values for *A* through *E* gives: $40 - \dfrac{3 \times 87 + \sqrt{291 - 2 \times 1} + 2 \times 1}{7}$

$= 40 - \dfrac{261 + \sqrt{289} + 2}{7} = 40 - \dfrac{261 + 17 + 2}{7}$

$40 - \dfrac{280}{7} = 40 - 40 = \mathbf{0}$.