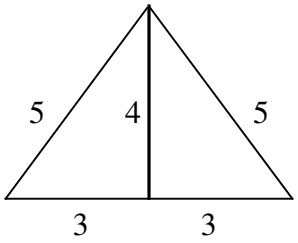
Intermediate Mathematics League of Eastern Massachusetts

Meet # 1
October, 2001

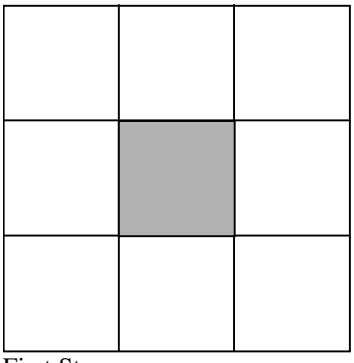
Category 1
Mystery
Meet #1, October, 2001

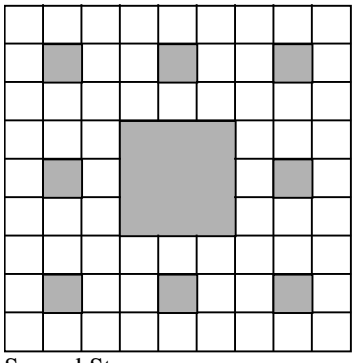
1. What number between 500 and 1000 is divisible by 5, 7, and 11?

2. How many distinct shapes can be made using two 3-4-5 triangles and placing them side by side so that they share an edge of the same length? One possible shape is shown below.



3. To create the first stage of the pattern below, the large square was divided into nine equal squares and the center square was shaded. To create the second stage, each white square was subdivided into nine equal squares and the center square was shaded. If the process is continued, what will be the total number of shaded squares in the third stage?


First Stage

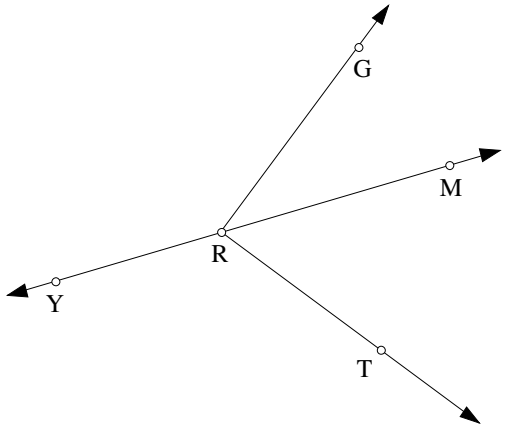

Second Stage

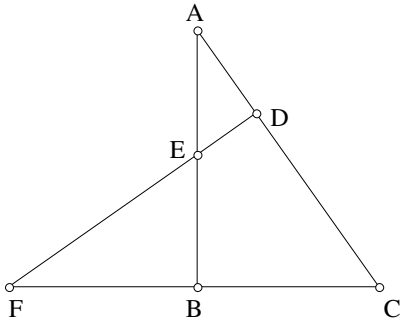
Answers

1. _____
2. _____
3. _____

Category 2
Geometry
Meet #1, October, 2001

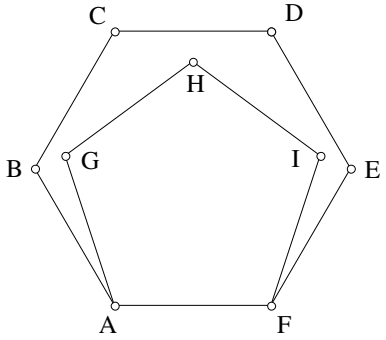
1. In the figure to the right, angles GRM and MRT are complementary. Angles MRT and TRY are supplementary. The measure of angle TRY is 127 degrees. How many degrees are in the measure of angle GRY?





2. In the figure to the left, angles ABC and ADE are right angles. FED and FBC are straight lines. The measure of angle ACB is 55 degrees. How many degrees are in the measure of angle EFB?

3. In the figure to the right, regular pentagon AGHIF sits inside regular hexagon ABCDEF so that the two shapes share base AF. How many degrees are in the measure of angle GAB?



Answers

1. _____
2. _____
3. _____

Category 3
Number Theory
Meet #1, October, 2001

1. What is the greatest possible difference of two primes if both of those primes are between 50 and 100?

2. Use the following clues to find the value of n:

- n is a whole number
- n is divisible by 24
- n is greater than 300
- n is not divisible by 17
- n is less than 400
- n is divisible by 21
- the sum of the digits of n is 12

3. Find the sum of all the *different* prime factors of 364,000.

Answers

1. _____
2. _____
3. _____

Category 4
Arithmetic
Meet #1, October, 2001

1. Jason and Clara found different decimal values for the following expression. Jason rounded each quotient to the nearest tenth and then added the results. Clara added the exact values of the quotients and then rounded the result to the nearest tenth. What is the positive difference between their answers?

$$15 \div 8 + 17 \div 20$$

2. Find the value of the expression

$$100 - 3\left\{6 + \left[7 \times 8 - 5(2+2) + 1\right] - 3 \times 6\right\}$$

3. Find the mean (average) value of the following expressions. Round your answer to the nearest whole number.

- $3^2 \times 4 + 5 =$
- $3 \times 4^2 + 5 =$
- $3 \times 4 + 5^2 =$

Answers
1. _____
2. _____
3. _____

Category 5
Algebra
Meet #1, October, 2001

1. Simplify: $\dfrac{6(2x+1)+3x+5(x-4)-4(3x+7)+18}{4}$

2. Find the value of z that makes the following equation an identity:

$$14x+8-2(3x+5)=(2x+3)-(z-6x)$$

3. Cindy plans to sell her car and she wants to get \$8000 for it. She knows that if a buyer makes an offer of a and she makes a counter-offer of b and if they negotiate back and forth, going exactly half way each time, the final price they settle on will be $\dfrac{a+2b}{3}$. If someone offers her \$6000, what should her counter-offer be, so that they settle on the \$8000 that she wants?

Answers

1. _____
2. _____
3. _____

Category 6
Team Questions
Meet #1, October, 2001

1. When x is divided by 11, the remainder is 3. What is the remainder when $4x$ is divided by 11?

2. The number in the middle square of the grid is the sum of some of the surrounding eight numbers. How many of those eight numbers contributed to this sum?

32	64	128
16	**107**	1
8	4	2

3. A man is to work for a year as a gardener on a large estate for $16,000 cash and a truck. He quits after 8 months and gets $8000 cash and the truck. How much is the truck worth?

4. Craig earned the following quiz scores for the first quarter marking period: 95, 83, 92, 89, 83, 92, 87, 83, 91, and 95. He gets to choose the mean, the median, or the mode of these scores for his grade. What is the value of the *greatest* of these three measures of central tendency?

5. The numbers in the pattern 1, 3, 6, 10, 15, etc. are known as triangular numbers. If the sum of three consecutive triangular numbers is 166, what is the second of these three numbers?

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

6. Using the values you obtained in questions 1 through 5, evaluate the following expression. Express your result in simplest terms.

$$\frac{\sqrt{\frac{C+10A}{D}+\frac{E}{B}}}{\frac{B+E}{A+B}}+\frac{B+E}{D}$$

Solutions to Category 1
Mystery
Meet #1, October, 2001

Answers

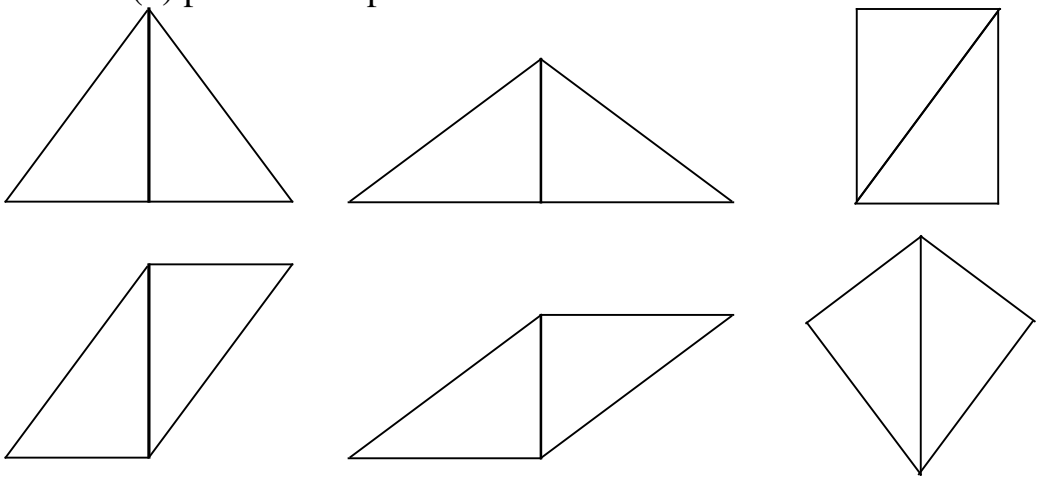
1. 770

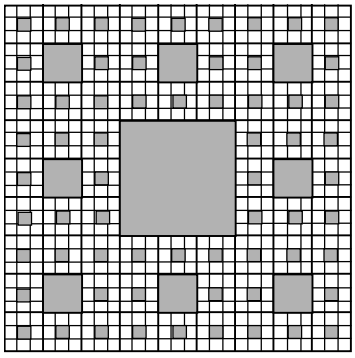
2. 6

3. 73

1. For a number to be divisible by 5, 7, and 11, it must be a multiple of 5, 7, and 11. Since these three numbers are relatively prime (no common factor greater than 1), their least common multiple is their product: $5 \times 7 \times 11 = 385$. This number is not greater than 500, however, so we must double it to get a common multiple that is between 500 and 1000: $2 \times 385 =$ **770**.

2. The six (**6**) possible shapes are shown below:





3. The third stage shown at left has $8^2 = 64$ little shaded squares, $8^1 = 8$ medium shaded squares, and $8^0 = 1$ large shaded square for a total of $64 + 8 + 1 =$ **73** shaded squares.

Solutions to Category 2
Geometry
Meet #1, October, 2001

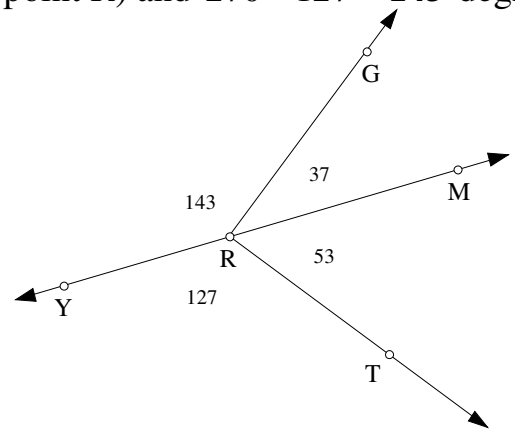
Answers

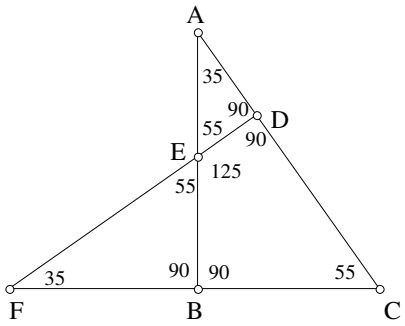
1. 143

2. 35

3. 12

1. We know that angle TRY measures 127, so its supplement, angle MRT, must be $180-127=53$ degrees. Angle GRM, the complement to angle MRT, must be $90-53=37$ degrees. Angle GRY is the supplement to GRM, so it must be $180-37=143$ degrees. Alternatively, we might notice that angles GRY and TRY must have a sum of 270 degrees (GRM and MRT together make the other 90 degrees in the 360 degrees around point R) and $270-127=\mathbf{143}$ degrees.





2. The figure contains four similar right triangles: ABC, ADE, FBE, and FDC. All four of these triangles have an angle that measures 90 degrees and an angle that measures 55 degrees. The third angle must equal 35 degrees since the total angle sum of any triangle is 180 degrees and $90+55+35=180$. In particular, triangle FDC contains a right angle at D and a 55 degree angle at C, so the angle at F must be **35** degrees.

3. Regular hexagons have interior angles of 120 degrees and regular pentagons have interior angles of 108 degrees. This can be determined by partitioning the polygon into triangles, each containing 180 degrees. Thus, the measure of angle GAB is $120-108=\mathbf{12}$ degrees.

Solutions to Category 3
Number Theory
Meet #1, October, 2001

Answers

1. 44

2. 336

3. 27

1. To get the greatest possible difference, we will want to find the prime closest to 100 and subtract the prime closest to 50. $99 = 9 \times 11$, so it's not prime, and $51 = 3 \times 17$, so it's not prime either. Both 97 and 53 are prime, though, so our greatest difference is $97 - 53$, which is **44**.

2. That *n* is divisible by 24 and 21 are the most helpful clues to start with. Since $24 = 2^3 \times 3$ and $21 = 3 \times 7$, we know that *n* is divisible by 2^3, 3, and 7, which are relatively prime. This means that *n* is a multiple of $2^3 \times 3 \times 7 = 168$. The only multiple of 168 that is between 300 and 400 is $2 \times 168 = 336$, so that must be it. Checking the other clues, we know that 336 cannot be divisible by 17, since we created it by multiplying $2 \times \left(2^3 \times 3 \times 7\right)$ and 17 is a different prime. The final clue says that the sum of the digits is 12 and this is true of our answer **336**.

3. $364{,}000 = 364 \times 1000$ and we know that $1000 = 2^3 \times 5^3$. Now we need to look for other prime factors in 364:
$364 = 2 \times 182 = 2^2 \times 91 = 2^2 \times 7 \times 13$. So the *different* prime factors of 364,000 are 2, 5, 7, and 13. The sum of these numbers is **27**.

Solutions to Category 4
Arithmetic
Meet #1, October, 2001

Answers

1. 0.1

2. 25

3. 44

1. Jason would round $15 \div 8 = 1.875$ to 1.9 and $17 \div 20 = 0.85$ to 0.9, obtaining a sum of 2.8. Clara would round the sum $1.875 + 0.85 = 2.725$ to 2.7. The positive difference between their answers is $2.8 - 2.7 = \mathbf{0.1}$.

2. Simplifying the expression a few steps at a time, according to the order of operations, we get:

$$100 - 3\{6 + [7 \times 8 - 5(2+2) + 1] - 3 \times 6\}$$
$$100 - 3\{6 + [56 - 20 + 1] - 18\}$$
$$100 - 3\{6 + [37] - 18\}$$
$$100 - 3\{25\}$$
$$100 - 75 = \mathbf{25}$$

3. First we need to find the value of each expression.

- $3^2 \times 4 + 5 = 9 \times 4 + 5 = 36 + 5 = 41$
- $3 \times 4^2 + 5 = 3 \times 16 + 5 = 48 + 5 = 53$
- $3 \times 4 + 5^2 = 12 + 25 = 37$

Now, to find the mean we add up these values and divide by 3. $41 + 53 + 37 = 131$ and $131 \div 3 = 43.\overline{6}$, which rounds to **44**.

Solutions to Category 5
Algebra
Meet #1, October, 2001

Answers

1. $2x - 6$

2. $z = 5$

3. \$9000

1. To simplify the expression, we must first use the distributive property to eliminate the parentheses as follows:

$$\frac{6(2x+1)+3x+5(x-4)-4(3x+7)+18}{4}$$

$$\frac{12x+6+3x+5x-20-12x-28+18}{4}$$

Now, combining like terms, we get:

$$\frac{8x-24}{4} = \frac{8x}{4} - \frac{24}{4} = \mathbf{2x - 6}$$

2. First we simplify each side of the equation:

$$14x + 8 - 2(3x+5) = (2x+3) - (z - 6x)$$

$$14x + 8 - 6x - 10 = 2x + 3 - z + 6x$$

$$8x - 2 = 8x + 3 - z$$

Since $8x = 8x$ for any value of x, this equation will be an identity when $-2 = 3 - z$, which happens only when $z = \mathbf{5}$.

3. We must solve the following equation for b, the counter offer:

$$\frac{6000 + 2b}{3} = 8000$$

Multiplying both sides by 3, we get:

$$6000 + 2b = 24{,}000$$

$$2b = 18{,}000$$

$$b = 9000$$

Cindy should make a counter offer of **\$9000**.

Solutions to Category 6
Algebra
Meet #1, October, 2001

Answers

1. 1

2. 5

3. $8000

4. 90

5. 55

6. $1\frac{2}{3}$

1. When $4x$ is divided by 11, this remainder would be four times the remainder when x is divided by 11 except that $4 \times 3 = 12$ is greater than 11. This means that one more 11 can be taken away and the remainder is **1**. Students can simply pick a number that leaves a remainder of 3 when divided by 11, such as 14, and divide 4 times that number by 11. If $x = 14$, then $4 \times 14 = 56$ and $56 \div 11$ leaves a remainder of **1**.

2. The only way to get a sum of 107 from the powers of 2 shown in the surrounding cells is to add $64 + 32 + 8 + 2 + 1$, which uses five (**5**) of the eight numbers.

3. The man received $8000 cash and all of the truck for eight months' work. The other $8000 cash that he did *not* receive must be what the other four months of work would have been worth. That means he earns $2000 every month or $24,000 for a year. Since he was to get $16,000 cash and the truck, the truck must be worth **$8000**.
Using algebra, one could write and solve the following equation:

$$\frac{2}{3}(16{,}000 + T) = 8000 + T$$

Tripling both sides to get rid of the fraction, we get:

$$2(16{,}000 + T) = 3(8000 + T)$$
$$32{,}000 + 2T = 24{,}000 + 3T$$
$$32{,}000 = 24{,}000 + T$$
$$8000 = T$$

4. The sum of Craig's quiz scores is 890 and $890 \div 10 = 89$, so the mean is 89. To find the median, we have to order the scores from least to greatest: 83, 83, 83, 87, 89, 91, 92, 92, 95, 95. Since there are an even number of scores, there is no clear middle, so we average the two middle scores and get 90 for the median. The mode is 83, since there are more 83's than any other score. For Craig, the greatest of these three measures of central tendency is the median, which is **90**.

5. The solution can be found through trial and error by continuing the list of triangular numbers: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, etc. The sum of the first three is 10, the sum of the next three is 19, etc. Eventually, we find that $45 + 55 + 66 = 166$. The second of these three numbers is **55**.

To solve this problem in general we can write the equation $(n - m) + n + (n + m + 1) = 166$, where n is the second of the triangular numbers (the one we want to find), the difference $(n - m)$ is the first triangular number (an unknown amount m less than n), and the third triangular number is $(n + m + 1)$. Simplifying our equation, we get:

$$3n + 1 = 166$$
$$3n = 165$$
$$n = \mathbf{55}$$

6. $$\frac{\sqrt{\frac{C+10A}{D}+\frac{E}{B}}}{\frac{B+E}{A+B}}+\frac{B+E}{D}=\frac{\sqrt{\frac{8000+10\times 1}{90}+\frac{55}{5}}}{\frac{5+55}{1+5}}+\frac{5+55}{90}$$

$$=\frac{\sqrt{89+11}}{10}+\frac{60}{90}=\frac{10}{10}+\frac{2}{3}=\mathbf{1\frac{2}{3}}.$$